Solutions for module 8

MCMC: Invariant density, irreducibility, Metropolis-Hastings algorithm.

Exercise 1

1. If $X^{(t)} \sim \pi(x)$ it must hold that

$$
\begin{aligned}
P(X^{(t+1)} \in A) &= \int P(X^{(t+1)} \in A | X^{(t)} = x)\pi(x)dx \\
&= \int P(x, A)\pi(x)dx = \int_A \pi(x)dx
\end{aligned}
$$

where the first equality follows from the total law of probability, the second from the definition of a transition kernel, and the last from the definition of an invariant density.

2. The answer follows by induction: From question 1 we have that, $X^{(t+1)} \sim \pi(x)$ if $X^{(t)} \sim \pi(x)$, and this implies that $X^{(t+2)} \sim \pi(x)$ if $X^{(t)} \sim \pi(x)$, and so on.

Exercise 2

1. By Exercise 1, since $X^{(0)} \sim \pi(x)$ we have that $X^{(t)} \sim \pi(x)$ for $t = 0, 1, 2, \ldots$. Hence $\mathbf{E}[h(X^{(t)})] = \mu$ for $t = 0, 1, 2, \ldots$. Then, using the usual rules for expectation, we obtain

$$
\mathbf{E}[\hat{\mu}_n] = \mathbf{E}\left[\frac{1}{n+1}\sum_{t=m}^{m+n} h(X^{(t)})\right] = \frac{1}{n+1}\sum_{t=m}^{m+n} \mathbf{E}[h(X^{(t)})] = \frac{1}{n+1}(n+1)\mu = \mu.
$$

Exercise 3

1. The proposal is uniformly distributed in the interval $[-x-\delta; -x+\delta]$. That is, if x is inside the "positive box" (i.e. $0.5 \leq x \leq 1.5$), then the interval $[-x-\delta; -x+\delta]$ intersects the "negative box" $(-1.5 \leq x \leq -0.5)$.

2. Assuming that x is inside the "positive or negative box", i.e. that $\pi(x) > 0$, then the acceptance probability is $1[|y+1| \leq \frac{1}{2}] + 1[|y-1| \leq \frac{1}{2}]$. This implies that any proposal inside any of the two boxes will be accepted.

3. The Markov chain is irreducible for all $\delta > 0$.

4. If x is sufficiently close to the sides of the two boxes there is a positive probability that the proposal y will be outside the two boxes and hence is rejected. This implies that the Markov chain is aperiodic.

Exercise 4

1. This Markov chain alternates between 1 and 0.

2. Consider the definition for invariant distribution in the case where $y = 0$:

$$
\begin{aligned}
&\pi(0)P(0, \{0\}) + \pi(1)P(1, \{0\}) = \pi(0) \\
\Rightarrow \quad &\pi(0) \cdot 0 + \pi(1) \cdot 1 = \pi(0) \\
\Rightarrow \quad &\pi(1) = \pi(0).
\end{aligned}
$$

As $\pi(0) + \pi(1) = 1$, the solution is $\pi(0) = \pi(1) = \frac{1}{2}$.

3. We have

$$
P(X^{(t)} = 0 | X^{(0)} = 0) = \begin{cases} 1 & \text{if } t = 2, 4, 6, \ldots \\ 0 & \text{otherwise} \end{cases}
$$

and

$$P(X^{(t)} = 1 | X^{(0)} = 0) = \begin{cases} 1 & \text{if } t = 1, 3, 5, \dots \\ 0 & \text{otherwise,} \end{cases}$$

so the Markov chain does not converge. As π is an invariant density and the Markov chain is irreducible, we still have that the law of large numbers holds. Hence, the Markov chain can be used for estimating expectations by Monte Carlo estimates.