Exercises for module 7

Markov chain Monte Carlo methods

Exercise 1

Consider the Metropolis-Hasthings (MH) algorithm, assuming the target density is a standard normal:

$$\pi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2).$$

Furthermore, assume that the proposals are normally distributed centred at the current value and with standard deviation σ:

$$q(x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y - x)^2\right).$$

1. Determine the acceptance probability.

The following is a pseudo code for implementing the MH algorithm:

choose inital value $x^{(0)}$
for i = *1,...,n* **do**
  generate proposal $y \sim q(x^{(i-1)}, y)$
  generate $u \sim \text{Unif}[0, 1]$
  calculate $H(x^{(i-1)}, y) = (\pi(y)q(y, x^{(i-1)}))/(\pi(x^{(i-1)})q(x^{(i-1)}, y))$
  if $u < H(x^{(i-1)}, y)$ **then**
    set $x^{(i)} = y$
  else
    set $x^{(i)} = x^{(i-1)}$
  end
end

2. Argue why in the MH algorithm the proposal y is accepted with probability $\min\{1, H(x, y)\}$.
3. Implement the code in `R` in such a way that it returns a realisation of the Markov chain. Make a trace plot and a histogram of the output. How large does n (i.e. the length of the chain) have to be for the histogram to look like a standard normal distribution. What effect does the value of σ have?
4. Use your code to estimate the probability $P(X \leq -1)$. The correct value is obtain in `R` by `pnorm(-1)`.
5. Assume now that the proposal kernel is

$$q(x, y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2).$$

   What is the interpretation of this proposal kernel? Find the resulting acceptance probability.