Exercises for module 6

The Gibbs sampler

1 IQ test

Suppose that n people have taken an IQ-test. The score x_i obtained by the ith person is assumed to be normally distributed with known precision τ and a mean μ_i which corresponds to the true IQ for that person, i.e. $x_i \sim N(\mu_i, \tau)$. Assume that the people taking the test come from a population where the true IQ can be assumed to be normally distributed, i.e. $\mu_i \sim N(\mu_G, \tau_G)$. Regarding priors we assume a priori that μ_G and τ_G are independent, and $\mu_G \sim N(\mu_0, \tau_0)$ and $\tau_G \sim \text{Gamma}(\alpha, \beta)$.

1. Determine the joint distribution $\pi(x_1, \ldots, x_n, \mu_1, \ldots, \mu_n, \mu_G, \tau_G)$.
2. Determine the full conditionals, $\pi(\mu_1 | \mu_2, \ldots, \mu_n, \mu_G, \tau_G, x_1, \ldots, x_n)$ etc.
   Hint: Have a look at the results for the case of n independent samples $x_1, \ldots, x_n$ from the same normal distributon $N(\mu, \tau)$.
3. Specify a Gibbs sampler for sampling $\pi(\mu_1, \ldots, \mu_n, \mu_G, \tau_G | x_1, \ldots, x_n)$.

2 Radiocarbon dating

Following Lee (2003, p. 263) consider the following example from archaeology: Assume that for each of three samples we have measured the date as x_1, x_2 and x_3 and a reasonable approximation is $x_i \sim N(\mu_i, \tau_i)$, where μ_i is the true age and τ_i is known. It is further known that the age of the samples are positive and below some upper limit k. In addition the time order of the three samples is known, that is $\mu_1 < \mu_2 < \mu_3$. Thus, as a joint prior for (μ_1, μ_2, μ_3) we propose

$$\pi(\mu_1, \mu_2, \mu_3) \propto \begin{cases} c & \text{if } 0 < \mu_1 < \mu_2 < \mu_3 < k \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where c is a positive constant.

1. Determine the joint posterior pdf for the three mean values, i.e. $\pi(\mu_1, \mu_2, \mu_3 | x_1, x_2, x_3)$.
2. Determine the full conditionals, i.e. $\pi(\mu_1 | \mu_2, \mu_3, x_1, x_2, x_3)$ etc.
   Notice: These distributions are non-standard.
3. Specify a Gibbs sampler for sampling of the posterior. How would you generate samples from the non-standard distributions above?