Solutions for module 1

Basics of probability theory

Exercise 1

1. Ω is the set of all sequences of length n where each element is either a H (head) or T (tail) (e.g. $HHTHTTTH$ if $n = 8$). That the coin is fair means that there is the same probability for observing H or T in a coin toss. Assuming that the n coin tosses are independent, P is specified by that $P(x) = 2^{-n}$ for any $x \in \Omega$. In other words, P is the uniform distribution on Ω.

2. Since
$$A^c = \{HH \ldots H,\ TT \ldots T\},$$
$P(A) = 1 - 2 \times 2^{-n}$. The event
$$B = \{HH \ldots H,\ TH \ldots H,\ HTH \ldots H,\ \ldots,\ HH \ldots HT\}$$
consists of $n+1$ states (or elements), so $P(B) = (n+1) \times 2^{-n}$. Finally,
$$A \cap B = \{TH \ldots H,\ HTH \ldots H,\ \ldots,\ HH \ldots HT\}$$
consists of n states, so $P(A \cap B) = n \times 2^{-n}$.

3. Since
$$P(A) \times P(B) = (n+1)(1 - 2^{1-n})2^{-n}$$
we obtain (the somewhat surprising) conclusion that A and B are independent if and only if $n = 3$:
$$n \times 2^{-n} = (n+1)(1 - 2^{1-n})2^{-n} \;\Leftrightarrow\; (n+1)2^{1-n} = 1 \;\Leftrightarrow\; n = 3.$$

Exercise 2

1.
$$F_X(x) = 0 \text{ if } x < 0, \qquad F_X(x) = x \text{ if } x \in [0,1], \qquad F_X(x) = 1 \text{ if } x > 1,$$
and so
$$f_X(x) = 1 \text{ if } x \in [0,1], \qquad f_X(x) = 0 \text{ otherwise.}$$

Hence

$$EX = \int_0^1 x dx = 1/2,\ E(X^2) = \int_0^1 x^2 dx = 1/3,\ Var(X) = 1/3 - (1/2)^2 = 1/12.$$

2.

$$P(\text{first decimal of } X \text{ is equal to } 1) = P(0.1 \leq X < 0.2) = 0.2 - 0.1 = 0.1.$$

Exercise 3

1. $F_X(x) = 0$ if $x < 0$, whilst for $x \geq 0$ we have that

$$F_X(x) = \int_0^x \lambda \exp(-\lambda x) dx = 1 - \exp(-\lambda x).$$

Furthermore, the mean is (use integration by parts):

$$E(X) = \int_0^\infty x\lambda \exp(-\lambda x) dx = ... = \frac{1}{\lambda}.$$

2.

$$P(X > t + s | X > s) = \frac{P(X > t + s)}{P(X > s)} = \frac{\exp(-\lambda(s + t))}{\exp(-\lambda s)} = \exp(-\lambda t)$$

so

$$P(X > t + s | X > s) = P(X > t)$$

does not depend on s, which can be interpret as follows: the exponential distribution (or equivalently X) has no memory.