Solutions for module 6
The Gibbs sampler

1 IQ test

1. The joint distribution of $(\underline{x}, \underline{\mu}, \mu_G, \tau_G)$ has pdf

$$
\begin{aligned}
\pi(\underline{x}, \underline{\mu}, \mu_G, \tau_G) &= \pi(\underline{x}|\underline{\mu}, \mu_G, \tau_G)\pi(\underline{\mu}, \mu_G, \tau_G) \\
&= \pi(\underline{x}|\underline{\mu}, \mu_G, \tau_G)\pi(\underline{\mu}|\mu_G, \tau_G)\pi(\mu_G, \tau_G) \\
&= \left(\prod_{i=1}^{n}\pi(x_i|\mu_i)\right)\left(\prod_{i=1}^{n}\pi(\mu_i|\mu_G, \tau_G)\right)\pi(\mu_G)\pi(\tau_G) \\
&= \left(\prod_{i=1}^{n}\sqrt{\frac{\tau_i}{2\pi}}\exp(-\frac{1}{2}\tau_i(x_i - \mu_i)^2)\right)\left(\prod_{i=1}^{n}\sqrt{\frac{\tau_G}{2\pi}}\exp(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2)\right) \times \\
&\qquad \left(\sqrt{\frac{\tau_0}{2\pi}}\exp(-\frac{1}{2}\tau_0(\mu_G - \mu_0)^2)\right)\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\tau_G^{\alpha-1}e^{-\tau_G/\beta}\right).
\end{aligned}
$$

2. The pdf for the full conditional distribution of μ_i is

$$
\begin{aligned}
\pi(\mu_i|\mu_1, \ldots, &\mu_{i-1}, \mu_{i+1}, \ldots, \mu_n, \mu_G, \tau_G, \underline{x}) \\
&\propto \pi(x_i|\mu_i)\pi(\mu_i|\mu_G, \tau_G) \\
&= \sqrt{\frac{\tau_i}{2\pi}}\exp\left(-\frac{1}{2}\tau_i(x_i - \mu_i)^2\right)\sqrt{\frac{\tau_G}{2\pi}}\exp\left(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2\right) \\
&\propto \exp\left(-\frac{1}{2}(\tau_i + \tau_G)\mu_i^2 + (\tau_i x_i + \tau_G\mu_G)\mu_i\right),
\end{aligned}
$$

which we recognise as the unnormalised density of a normal distributed random variable with mean $(\tau_i x_i + \tau_G\mu_G)/(\tau_i + \tau_G)$ and precision $(\tau_i + \tau_G)$. Hence the full conditional distribution of μ_i is

$$
\mu_i|\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n, \mu_g\tau_g, \underline{x} \sim N\left(\frac{\tau_i x_i + \tau_G\mu_G}{\tau_i + \tau_G}, \tau_i + \tau_G\right).
$$

The pdf for the full conditional distribution of μ_G is

$$
\begin{aligned}
\pi(\mu_G|\underline{\mu}, \tau_G, \underline{x}) &\propto \prod_{i=1}^{n}\pi(\mu_i|\mu_G, \tau_G)\pi(\mu_G) \\
&= \left(\prod_{i=1}^{n}\sqrt{\frac{\tau_G}{2\pi}}\exp\left(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2\right)\right)\sqrt{\frac{\tau_0}{2\pi}}\exp\left(-\frac{1}{2}\tau_0(\mu_G - \mu_0)^2\right) \\
&\propto \exp\left(-\frac{1}{2}\tau_G\sum(\mu_i - \mu_G)^2\right)\exp\left(-\frac{1}{2}\tau_0(\mu_G - \mu_0)^2\right) \\
&= \exp\left(-\frac{1}{2}(n\tau_G + \tau_0)\mu_G^2 + (\tau_G n\bar{\mu} + \tau_o\mu_0)\mu_G - \frac{1}{2}(\tau_G + \tau_0)\sum\mu_i^2\right).
\end{aligned}
$$

Again we recognise this an unnormalised density of a normal distributed random variable obtaining

$$
\mu_G|\underline{\mu}, \tau_G, \underline{x} \sim N\left(\frac{n\tau_G\bar{\mu} + \tau_o\mu_0}{n\tau_G + \tau_0}, n\tau_G + \tau_0\right).
$$

Finally, the full conditional distribution for τ_G is given by

$$\pi(\tau_G|\underline{\mu},\mu_g,\underline{x}) \propto \prod_{i=1}^{n} \pi(\mu_i|\mu_G,\tau_G)\pi(\tau_G)$$

$$= \left(\prod_{i=1}^{n}\sqrt{\frac{\tau_G}{2\pi}}\exp(-\frac{1}{2}\tau_G(\mu_i-\mu_G)^2)\right)\tau_G^{\alpha-1}\beta^{\alpha}e^{-\tau_G/\beta}$$

$$\propto \tau_G^{n/2+\alpha-1}\exp\left(-\tau_G\left(\frac{1}{2}\sum_{i=1}^{n}(\mu_i-\mu_G)^2+1/\beta\right)\right),$$

which can be recognised as the unnormalised denisty of a gamma distributted random variable, hence

$$\tau_G|\underline{\mu},\mu_g,\underline{x} \sim Gamma\left(\frac{n}{2}+\alpha,\left(\frac{1}{2}\sum_{i=1}^{n}(\mu_i-\mu_G)^2+1/\beta\right)^{-1}\right).$$

3. First step in a Gibbs sampler is to chose inital values. For $\mu_1^{(0)},\ldots,\mu_n^{(0)}$ and μ_G any values will, in principle, do. The initial value $\tau_G^{(0)}$ should be positive. A more "clever" choice could be to set $\mu_i^{(0)}=x_i$ for $i=1,\ldots,n$, $\mu_G^{(0)}=\bar{x}\equiv\frac{1}{n}\sum_{i=1}^{n}x_i$ and $\tau_G^{(0)}=n(\sum_{i=1}^{n}(x_i-\bar{x})^2)^{-1}$.

A Gibbs sampler could then proceed as follows

```
For t = 1 ...T do
```
  - `For i = 1 ...n do`
    - `Generate` $\mu_i^{(t)}$ `from` $N\left(\frac{\tau x_i+\tau_G^{(t-1)}\mu_G^{(t-1)}}{\tau^{(t-1)}+\tau_G^{(t-1)}},\tau^{(t-1)}+\tau_G^{(t-1)}\right)$
  - `Generate` $\mu_G^{(t)}$ `from` $N\left(\frac{n\tau_G^{(t-1)}\bar{\mu}^{(t)}+\tau_0\mu_0}{n\tau_G^{(t-1)}+\tau_0},n\tau_G^{(t-1)}+\tau_0\right)$
  - `Generate` $\tau_G^{(t)}$ `from` $Gamma\left(\frac{n}{2}+\alpha,\left(\frac{1}{2}\sum_{i=1}^{n}(\mu_i^{(t)}-\mu_G^{(t)})^2+1/\beta\right)^{-1}\right)$

2 Radiocarbon dating

Note that
$$\pi(\mu_1,\mu_2,\mu_3)\propto \mathbf{1}[0<\mu_1]\mathbf{1}[\mu_1<\mu_2]\mathbf{1}[\mu_2<\mu_3]\mathbf{1}[\mu_3<k],$$
where $\mathbf{1}[\cdot]$ is the indicator function:

$$\mathbf{1}["expression"]=\begin{cases}1 & \text{if }"expression"\text{ is true}\\0 & \text{otherwise.}\end{cases}$$

1. The joint posterior pdf is given by

$$\pi(\mu_1,\mu_2,\mu_3|x_1,x_2,x_3)\propto \pi(x_1,x_2,x_3|\mu_1,\mu_2,\mu_3)\pi(\mu_1,\mu_2,\mu_3)$$

$$=\pi(x_1|\mu_1)\pi(x_2|\mu_2)\pi(x_3|\mu_3)\pi(\mu_1,\mu_2,\mu_3)$$

$$=\left(\prod_{i=1}^{3}\sqrt{\frac{\tau}{2\pi}}\exp(-\frac{1}{2}\tau(x_i-\mu_i)^2)\right)\times$$

$$\mathbf{1}[0<\mu_1]\mathbf{1}[\mu_1<\mu_2]\mathbf{1}[\mu_2<\mu_3]\mathbf{1}[\mu_3<k].$$

2. The full conditional for μ_1 has pdf

$$\pi(\mu_1|\mu_2,\mu_3,x_1,x_2,x_3)\propto \pi(x_1|\mu_1)p(\mu_1,\mu_2,\mu_3)$$

$$=\sqrt{\frac{\tau}{2\pi}}\exp(-\frac{1}{2}\tau(x_1-\mu_1)^2)\times$$

$$\mathbf{1}[0<\mu_1]\mathbf{1}[\mu_1<\mu_2]\mathbf{1}[\mu_2<\mu_3]\mathbf{1}[\mu_3<k]$$

$$\propto \exp\left(-\frac{1}{2}\tau(x_1-\mu_1)^2\right)\mathbf{1}[0<\mu_1<\mu_2],$$

which is the density of a normal distribution $N(x_1, \tau)$ restricted to the open interval $(0, \mu_2)$. In a similar way we show that $\pi(\mu_2|\mu_1, \mu_3, x_1, x_2, x_3)$ is the density of $N(x_2, \tau)$ restricted to (μ_1, μ_3) and that $\pi(\mu_3|\mu_1, \mu_2, x_1, x_2, x_3)$ is the density of $N(x_3, \tau)$ restricted to (μ_2, k).

3. To generate a sample μ_1 from $\pi(\mu_1|\mu_2, \mu_3, x_1, x_2, x_3)$ we proceeds as follows: Generate proposals μ_1 from $N(x_1, \tau)$ until $\mu_1 \in (0, \mu_2)$. When this happens return μ_1 as a sample from $\pi(\mu_1|\mu_2, \mu_3, x_1, x_2, x_3)$. This procedure is an example of so-called rejection sampling.