

# Solutions for module 6

## The Gibbs sampler

### 1 IQ test

1. The joint distribution of  $(\underline{x}, \underline{\mu}, \mu_G, \tau_G)$  has pdf

$$\begin{aligned}
 \pi(\underline{x}, \underline{\mu}, \mu_G, \tau_G) &= \pi(\underline{x}|\underline{\mu}, \mu_G, \tau_G)\pi(\underline{\mu}, \mu_G, \tau_G) \\
 &= \pi(\underline{x}|\underline{\mu}, \mu_G, \tau_G)\pi(\underline{\mu}|\mu_G, \tau_G)\pi(\mu_G, \tau_G) \\
 &= \left( \prod_{i=1}^n \pi(x_i|\mu_i) \right) \left( \prod_{i=1}^n \pi(\mu_i|\mu_G, \tau_G) \right) \pi(\mu_G)\pi(\tau_G) \\
 &= \left( \prod_{i=1}^n \sqrt{\frac{\tau_i}{2\pi}} \exp\left(-\frac{1}{2}\tau_i(x_i - \mu_i)^2\right) \right) \left( \prod_{i=1}^n \sqrt{\frac{\tau_G}{2\pi}} \exp\left(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2\right) \right) \times \\
 &\quad \left( \sqrt{\frac{\tau_0}{2\pi}} \exp\left(-\frac{1}{2}\tau_0(\mu_G - \mu_0)^2\right) \right) \left( \frac{1}{\Gamma(\alpha)\beta^\alpha} \tau_G^{\alpha-1} e^{-\tau_G/\beta} \right).
 \end{aligned}$$

2. The pdf for the full conditional distribution of  $\mu_i$  is

$$\begin{aligned}
 \pi(\mu_i|\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n, \mu_G, \tau_G, \underline{x}) \\
 &\propto \pi(x_i|\mu_i)\pi(\mu_i|\mu_G, \tau_G) \\
 &= \sqrt{\frac{\tau_i}{2\pi}} \exp\left(-\frac{1}{2}\tau_i(x_i - \mu_i)^2\right) \sqrt{\frac{\tau_G}{2\pi}} \exp\left(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2\right) \\
 &\propto \exp\left(-\frac{1}{2}(\tau_i + \tau_G)\mu_i^2 + (\tau_i x_i + \tau_G \mu_G)\mu_i\right),
 \end{aligned}$$

which we recognise as the unnormalised density of a normal distributed random variable with mean  $(\tau_i x_i + \tau_G \mu_G)/(\tau_i + \tau_G)$  and precision  $(\tau_i + \tau_G)$ . Hence the full conditional distribution of  $\mu_i$  is

$$\mu_i|\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n, \mu_G, \tau_G, \underline{x} \sim N\left(\frac{\tau_i x_i + \tau_G \mu_G}{\tau_i + \tau_G}, \tau_i + \tau_G\right).$$

The pdf for the full conditional distribution of  $\mu_G$  is

$$\begin{aligned}
 \pi(\mu_G|\underline{\mu}, \tau_G, \underline{x}) &\propto \prod_{i=1}^n \pi(\mu_i|\mu_G, \tau_G)\pi(\mu_G) \\
 &= \left( \prod_{i=1}^n \sqrt{\frac{\tau_G}{2\pi}} \exp\left(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2\right) \right) \sqrt{\frac{\tau_0}{2\pi}} \exp\left(-\frac{1}{2}\tau_0(\mu_G - \mu_0)^2\right) \\
 &\propto \exp\left(-\frac{1}{2}\tau_G \sum (\mu_i - \mu_G)^2\right) \exp\left(-\frac{1}{2}\tau_0(\mu_G - \mu_0)^2\right) \\
 &= \exp\left(-\frac{1}{2}(n\tau_G + \tau_0)\mu_G^2 + (\tau_G \bar{\mu} + \tau_0 \mu_0)\mu_G - \frac{1}{2}(\tau_G + \tau_0) \sum \mu_i^2\right).
 \end{aligned}$$

Again we recognise this an unnormalised density of a normal distributed random variable obtaining

$$\mu_G|\underline{\mu}, \tau_G, \underline{x} \sim N\left(\frac{n\tau_G \bar{\mu} + \tau_0 \mu_0}{n\tau_G + \tau_0}, n\tau_G + \tau_0\right).$$

Finally, the full conditional distribution for  $\tau_G$  is given by

$$\begin{aligned}
 \pi(\tau_G|\underline{\mu}, \mu_G, \underline{x}) &\propto \prod_{i=1}^n \pi(\mu_i|\mu_G, \tau_G)\pi(\tau_G) \\
 &= \left( \prod_{i=1}^n \sqrt{\frac{\tau_G}{2\pi}} \exp\left(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2\right) \right) \tau_G^{\alpha-1} \beta^\alpha e^{-\tau_G/\beta} \\
 &\propto \tau_G^{n/2+\alpha-1} \exp\left(-\tau_G\left(\frac{1}{2}\sum_{i=1}^n (\mu_i - \mu_G)^2 + 1/\beta\right)\right),
 \end{aligned}$$

which can be recognised as the unnormalised density of a gamma distributed random variable, hence

$$\tau_G | \underline{\mu}, \mu_g, \underline{x} \sim \text{Gamma} \left( \frac{n}{2} + \alpha, \left( \frac{1}{2} \sum_{i=1}^n (\mu_i - \mu_G)^2 + 1/\beta \right)^{-1} \right).$$

3. First step in a Gibbs sampler is to choose initial values. For  $\mu_1^{(0)}, \dots, \mu_n^{(0)}$  and  $\mu_G$  any values will, in principle, do. The initial value  $\tau_G^{(0)}$  should be positive. A more “clever” choice could be to set  $\mu_i^{(0)} = x_i$  for  $i = 1, \dots, n$ ,  $\mu_G^{(0)} = \bar{x} \equiv \frac{1}{n} \sum_{i=1}^n x_i$  and  $\tau_G^{(0)} = n(\sum_{i=1}^n (x_i - \bar{x})^2)^{-1}$ .

A Gibbs sampler could then proceed as follows

For  $t = 1 \dots T$  do

• For  $i = 1 \dots n$  do

• Generate  $\mu_i^{(t)}$  from  $N \left( \frac{\tau x_i + \tau_G^{(t-1)} \mu_G^{(t-1)}}{\tau^{(t-1)} + \tau_G^{(t-1)}}, \tau^{(t-1)} + \tau_G^{(t-1)} \right)$

• Generate  $\mu_G^{(t)}$  from  $N \left( \frac{n \tau_G^{(t-1)} \bar{\mu}^{(t)} + \tau_0 \mu_0}{n \tau_G^{(t-1)} + \tau_0}, n \tau_G^{(t-1)} + \tau_0 \right)$

• Generate  $\tau_G^{(t)}$  from  $\text{Gamma} \left( \frac{n}{2} + \alpha, \left( \frac{1}{2} \sum_{i=1}^n (\mu_i^{(t)} - \mu_G^{(t)})^2 + 1/\beta \right)^{-1} \right)$

## 2 Radiocarbon dating

Note that

$$\pi(\mu_1, \mu_2, \mu_3) \propto \mathbf{1}[0 < \mu_1] \mathbf{1}[\mu_1 < \mu_2] \mathbf{1}[\mu_2 < \mu_3] \mathbf{1}[\mu_3 < k],$$

where  $\mathbf{1}[\cdot]$  is the indicator function:

$$\mathbf{1}[\text{“expression”}] = \begin{cases} 1 & \text{if “expression” is true} \\ 0 & \text{otherwise.} \end{cases}$$

1. The joint posterior pdf is given by

$$\begin{aligned} \pi(\mu_1, \mu_2, \mu_3 | x_1, x_2, x_3) &\propto \pi(x_1, x_2, x_3 | \mu_1, \mu_2, \mu_3) \pi(\mu_1, \mu_2, \mu_3) \\ &= \pi(x_1 | \mu_1) \pi(x_2 | \mu_2) \pi(x_3 | \mu_3) \pi(\mu_1, \mu_2, \mu_3) \\ &= \left( \prod_{i=1}^3 \sqrt{\frac{\tau}{2\pi}} \exp\left(-\frac{1}{2} \tau (x_i - \mu_i)^2\right) \right) \times \\ &\quad \mathbf{1}[0 < \mu_1] \mathbf{1}[\mu_1 < \mu_2] \mathbf{1}[\mu_2 < \mu_3] \mathbf{1}[\mu_3 < k]. \end{aligned}$$

2. The full conditional for  $\mu_1$  has pdf

$$\begin{aligned} \pi(\mu_1 | \mu_2, \mu_3, x_1, x_2, x_3) &\propto \pi(x_1 | \mu_1) p(\mu_1, \mu_2, \mu_3) \\ &= \sqrt{\frac{\tau}{2\pi}} \exp\left(-\frac{1}{2} \tau (x_1 - \mu_1)^2\right) \times \\ &\quad \mathbf{1}[0 < \mu_1] \mathbf{1}[\mu_1 < \mu_2] \mathbf{1}[\mu_2 < \mu_3] \mathbf{1}[\mu_3 < k] \\ &\propto \exp\left(-\frac{1}{2} \tau (x_1 - \mu_1)^2\right) \mathbf{1}[0 < \mu_1 < \mu_2], \end{aligned}$$

which is the density of a normal distribution with mean  $x_1$  and precision  $\tau$  restricted to the open interval  $(0, \mu_2)$ . In similar fashion we can show that  $\pi(\mu_2 | \mu_1, \mu_3, x_1, x_2, x_3)$  and  $\pi(\mu_3 | \mu_1, \mu_2, x_1, x_2, x_3)$  correspond to normal densities restricted to some open interval.

3. To generate a sample  $\mu_1$  from  $\pi(\mu_1 | \mu_2, \mu_3, x_1, x_2, x_3)$  we proceed as follows: Generate proposals  $\mu_1$  from  $N(x_1, \tau)$  until  $\mu_1 \in (0, \mu_2)$ . When this happens return  $\mu_1$  as a sample from  $\pi(\mu_1 | \mu_2, \mu_3, x_1, x_2, x_3)$ . This procedure is an example of so-called rejection sampling.