Bayesian statistics, simulation and software

Module 10: Bayesian prediction and model checking

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Prior predictions

Suppose we want to predict future data $\tilde{x}$ *without* observing any data x.

Assume:

- **Data model**: $\tilde{x}|\theta \sim x|\theta \sim \pi(x|\theta)$.
- **Prior**: $\pi(\theta)$.

This implies a joint distribution:

$$(\tilde{x}, \theta) \sim \pi(x, \theta) = \pi(x|\theta)\pi(\theta).$$

From this joint distribution we obtain the marginal density of $\tilde{x}$,

$$\tilde{x} \sim \pi(\tilde{x}) = \int \pi(x|\theta)\pi(\theta) d\theta,$$

which is called the **prior predictive density**.

Prior prediction: Normal case, τ known

Assume:

- **Data model**: $\pi(x|\mu) \sim \mathcal{N}(\mu, \tau)$.
- **Prior**: $\pi(\mu) \sim \mathcal{N}(\mu_0, \tau_0)$.

Prior predictive density:

$$\begin{aligned}\pi(x) &= \int \pi(x|\mu)\pi(\mu) d\mu \\ &= \int \sqrt{\frac{\tau}{2\pi}} \exp\left(-\frac{1}{2}\tau(x-\mu)^2\right) \sqrt{\frac{\tau_0}{2\pi}} \exp\left(-\frac{1}{2}\tau_0(\mu-\mu_0)^2\right) d\mu\end{aligned}$$

and a simple calculation shows

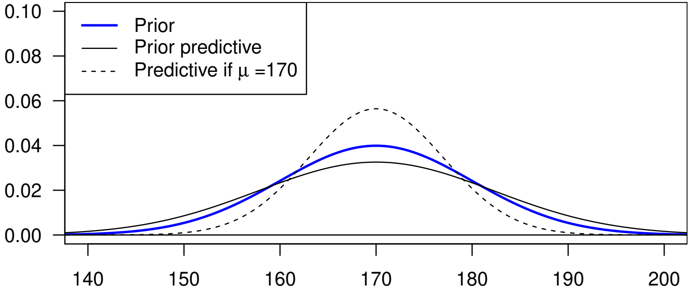
$$\pi(x) \propto \exp\left(-\frac{1}{2}\frac{\tau\tau_0}{\tau+\tau_0}(x-\mu_0)^2\right) \sim \mathcal{N}\left(\mu_0, \frac{\tau\tau_0}{\tau+\tau_0}\right).$$

Easier argument: $x-\mu \sim \mathcal{N}(0,\tau)$ is independent of $\mu \sim \mathcal{N}(\mu_0, \tau_0)$, so $x \sim \mathcal{N}\left(\mu_0, (\frac{1}{\tau}+\frac{1}{\tau_0})^{-1}\right) = \mathcal{N}\left(\mu_0, \frac{\tau\tau_0}{\tau+\tau_0}\right)$.

NB: prior predictive var. is larger than both prior var. and data var.

Prior predictive distribution

Illustration of the fact that prior predictive precision $<$ prior precision (ignore the dashed line):



Simulating the prior predictive distribution

If the prior predictive density $\pi(x)$ is difficult to derive we can simply make a simulation $\tilde{x}$ in two steps:

1. Generate parameter from prior: $\theta \sim \pi(\theta)$.
2. Conditional on θ generate $\tilde{x}$: $\tilde{x} \sim \pi(x|\theta)$.

Posterior prediction

Now, suppose we have observed data x and want to predict a possible *future* observation $\tilde{x}$ *given* data x.

Assume:

- **Data model**: $\tilde{x}|\theta \sim x|\theta \sim \pi(x|\theta)$, and given θ then x and $\tilde{x}$ are independent.
- **Prior**: $\pi(\theta)$.

The joint density of predicted data $\tilde{x}$, data x and parameter θ is

$$\begin{aligned}\pi(\tilde{x}, x, \theta) = \pi(\tilde{x}, x \mid \theta)\pi(\theta) &= \pi(\tilde{x}|\theta)\pi(x|\theta)\pi(\theta) \\ &= \pi(\tilde{x}|\theta)\pi(\theta|x)\pi(x)\end{aligned}$$

where $\pi(\tilde{x}|\theta)$ and $\pi(x|\theta)$ represent the same conditional distribution (namely that from the data model).

The **posterior predictive distribution** is the (marginal) distribution of $\tilde{x}$ conditional on data x:

$$\pi(\tilde{x}|x) = \int \pi(\tilde{x}, \theta|x)d\theta = \int \frac{\pi(\tilde{x}, \theta, x)}{\pi(x)} d\theta = \int \pi(\tilde{x}|\theta)\pi(\theta|x)d\theta.$$

Thus we have now replaced the prior density with the posterior density.

Simulating the posterior predictive distribution

If the posterior predictive density $\pi(\tilde{x}|x)$ is difficult to derive we can simply make a simulation $\tilde{x}$ in two steps:

1. Generate parameter from posterior: $\theta|x \sim \pi(\theta|x)$.
2. Conditional on θ generate $\tilde{x}$ from data model: $\tilde{x} \sim \pi(x|\theta)$.

Posterior prediction: Normal case, τ known

Data model: $X_1, X_2, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \tau)$.

Prior: $\pi(\mu) \sim \mathcal{N}(\mu_0, \tau_0)$.

Posterior: $\pi(\mu|\mathbf{x}) \sim \mathcal{N}(\mu_1, \tau_1)$, $\mu_1 = \frac{n\tau\bar{x}+\tau_0\mu_0}{n\tau+\tau_0}$ and $\tau_1 = n\tau + \tau_0$.

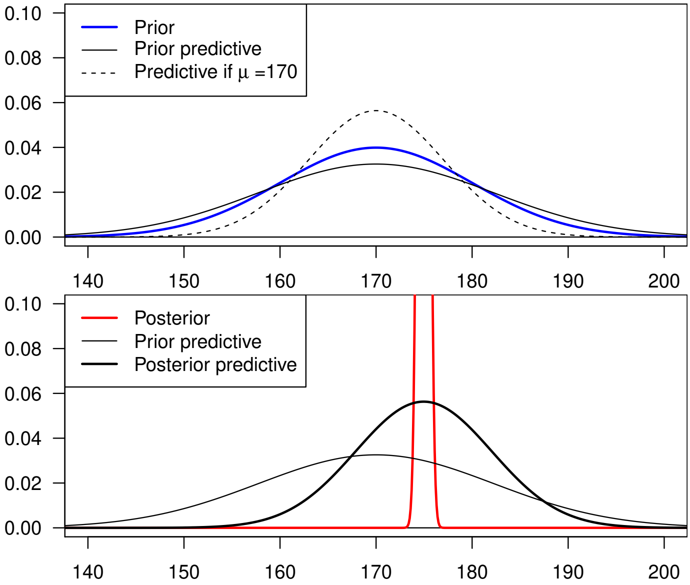
As the prior predictive distribution (of one observation) is $\mathcal{N}\left(\mu_0, \frac{\tau_0\tau}{\tau+\tau_0}\right)$ and the posterior is the "prior" for the posterior prediction, we obtain by replacing μ_0 by μ_1 and replacing τ_0 by τ_1 that

$$\tilde{x}|\mathbf{x} \sim \mathcal{N}\left(\mu_1, \frac{\tau\tau_1}{\tau+\tau_1}\right) = \mathcal{N}\left(\frac{n\tau\bar{x}+\tau_0\mu_0}{n\tau+\tau_0}, \frac{(n\tau+\tau_0)\tau}{\tau+n\tau+\tau_0}\right).$$

NB: Posterior predictive mean and posterior mean are equal but posterior predictive precision $\frac{\tau\tau_1}{\tau+\tau_1}$ is smaller than posterior precision τ_1 and smaller than prior precision τ. When n is large, we have $\tilde{x}|\mathbf{x} \overset{approx}{\sim} \mathcal{N}(\bar{x}, \tau)$.

When $\tau_0 = 0$ (i.e. we consider an improper prior), we have (as in classical statistics) $\mu|x \sim \mathcal{N}(\bar{x}, n\tau)$ and $\tilde{x}|\mathbf{x} \overset{approx}{\sim} \mathcal{N}(\bar{x}, \tau)$ (for n large).

Prior and posterior predictive distributions



Model checking

Idea: If the model is correct, then posterior predictions of the data should look like the observed data. **Difficulty**: How to choose a good measure of "similarity"?

Example: We have observed a sequence of $n = 20$ zeros and ones:

$$1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0$$

Model: $X_1, X_2, \ldots, X_{20}$ are IID where $P(X_i = 1) = p$ is unknown.

Prior: $\pi(p) \sim Be(\alpha, \beta)$ where $\alpha > 0$ and $\beta > 0$ are known.

Posterior: $\pi(p|\mathbf{x}) \sim Be(\#\text{ones} + \alpha, \#\text{zeros} + \beta)$.

Model checking: We simulate N posterior predictive realisations

$$\tilde{\mathbf{X}}^{(i)} = (\tilde{X}_1^{(i)}, \tilde{X}_2^{(i)}, \ldots, \tilde{X}_{20}^{(i)}) \quad i = 1, \ldots, N.$$

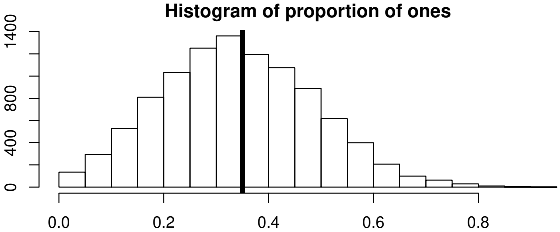
If these vectors look "similar" to the data above, it would indicate that the model is probably okay.

Model checking: First attempt (a failure)

Define summary function

$$s(\mathbf{x}) = \#\text{ones in } \mathbf{x}.$$

Histogram for $s(\tilde{\mathbf{x}}^{(i)})$ for $N = 10,000$ independent posterior predictions:



So the observed number of ones is in no way unusual compared to the posterior predictions.

This is just as expected — so we need another summary function $s(\mathbf{x})$.

Model checking: Second attempt (a success)

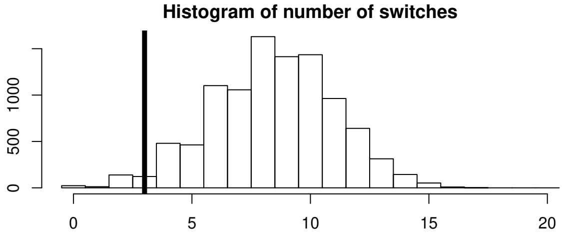
Define summary function

$$s(\mathbf{x}) = \text{number of switches between ones and zeros in } \mathbf{x}.$$

In the data the number of switches is 3:

$$1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0$$

Histogram for $s(\tilde{\mathbf{x}}^{(i)})$ for $N = 10,000$ independent posterior predictions:

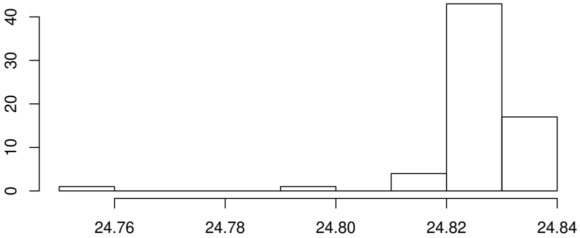


Only around 1.7% of the posterior prediction have 3 or fewer switches.

This suggests that the model assumption of independence is questionable.

Example: Speed of light

66 measurements of the time it takes light to travel 7445 meters (deviations in nanoseconds from a given number):



Data model:

$$x_1, \ldots, x_{66} \overset{iid}{\sim} \mathcal{N}(\mu, \tau).$$

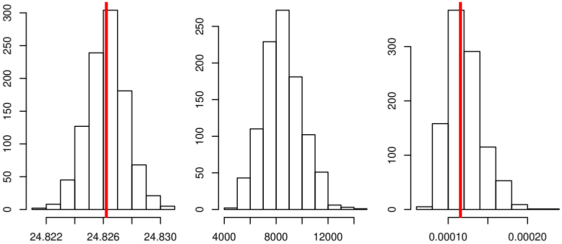
(Questionable?)

Prior:

$$\pi(\mu, \tau) \sim \mathcal{N}(0, 0.001) \times \text{Gamma}(0.001, 1000).$$

Example: Speed of light

Posterior distribution of μ, τ and $1/\tau$:



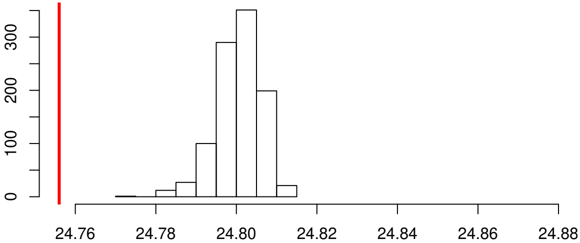
Red lines denote sample mean and sample variance, respectively.

Example: Speed of light

Data contain one very low measurement. Is this unusual?

Generate 1000 posterior predictive samples $\mathbf{x}^{(i)} = (x_1^{(i)}, \ldots, x_{66}^{(i)})$, $i = 1, \ldots, 1000$, and define

$$s(\mathbf{x}) = \min\{x_1, \ldots x_{66}\}.$$



Conclusion: The smallest value in the data is very unlikely under the assumed model.