Exercises for module 8

MCMC: Invariant density, irreducibility, Metropolis-Hastings algorithm.

Exercise 1

Assume the transition kernel $P(x, A)$ specifies a Markov chain with invariant density $\pi(x)$.

1. Show that if $X^{(t)} \sim \pi(x)$, then $X^{(t+1)} \sim \pi(x)$.
   Hint: You need to show that $P(X^{(t+1)} \in A) = \int_A \pi(x)dx$.
2. Argue that if $X^{(t)} \sim \pi(x)$, then $X^{(t+n)} \sim \pi(x)$ for all $n \geq 0$.

Exercise 2

Assume that $X^{(0)}, X^{(1)}, \ldots$ is an irreducible Markov chain with invariant density $\pi(x)$ and we are given a function h so that $\mu = \int h(x)\pi(x)dx$ exists. Recall the definition

$$\hat{\mu}_n = \frac{1}{n+1} \sum_{t=m}^{m+n} h(X^{(t)}).$$

1. Assuming that $X^{(0)} \sim \pi(x)$, show that $\hat{\mu}_n$ is an *unbiased* estimator of μ, i.e. show that $E[\hat{\mu}_n] = \mu$.

Exercise 3 Consider the "two box" target density from the slides:

$$\pi(x) = \frac{1}{2} \cdot 1 \left[|x+1| \leq \frac{1}{2} \right] + \frac{1}{2} \cdot 1 \left[|x-1| \leq \frac{1}{2} \right].$$

Furthermore, consider the Metropolis-Hastings algorithm with the following proposal kernel:

$$q(x, y) = \frac{1}{2\delta} \cdot 1 \left[|y + x| \leq \delta \right],$$

where $\delta > 0$.

1. What is the interpretation of the proposal kernel?
2. Determine the acceptance probability.
3. For what values of δ is the resulting Markov chain irreducible.
4. Why is the Markov chain aperiodic?

Exercise 4

Consider a Markov chain on a discrete state space $\Omega = \{0, 1\}$ with transition kernel given by

$$P(0, \{1\}) = P(1, \{0\}) = 1,$$
$$P(0, \{0\}) = P(1, \{1\}) = 0.$$

1. What does this (rather borring) Markov chain look like?

In the discrete case, the definition of an invariant density is

$$\sum_{x \in \Omega} \pi(x) P(x, \{y\}) = \pi(y) \qquad \text{for all } y \in \Omega.$$

2. Find the invariant distribution of this Markov chain.
3. What is the conditional distribution of $X^{(t)}$ for any $t > 0$ when $X^{(0)} = 0$?