Contingency tables

The ASTA team

Contents

1 Contingency tables

1.1 A contingency table

- We return to the dataset `popularKids`, where we study **association** between 2 **factors**: `Goals` and `Urban.Rural`.
- Based on a sample we make a cross tabulation of the factors and we get a so-called **contingency table** (*krydstabel*).

```
import pandas as pd

popKids = pd.read_csv("https://asta.math.aau.dk/datasets?file=PopularKids.dat", sep="\t")
popKids = popKids.rename(columns={'Urban/Rural': 'Urban.Rural'})
```

```
tab_totals = pd.crosstab(popKids['Urban.Rural'], popKids['Goals'], margins=True)
tab_totals
```

```
## Goals        Grades  Popular  Sports  All
## Urban.Rural
## Rural            57       50      42  149
## Suburban         87       42      22  151
## Urban           103       49      26  178
## All             247      141      90  478
```

1.1.1 A conditional distribution

- Another representation of data is the percent-wise distribution of `Goals` for each level of `Urban.Rural`, i.e. the sum in each row of the table is 100 (up to rounding):

```
tab = pd.crosstab(popKids['Urban.Rural'], popKids['Goals'], margins=False)
tab_pct = (tab.div(tab.sum(axis=1), axis=0) * 100)
tab_pct['All'] = tab_pct.sum(axis=1)
tab_pct.round()
```

```
## Goals        Grades  Popular  Sports    All
## Urban.Rural
## Rural          38.0     34.0    28.0  100.0
## Suburban       58.0     28.0    15.0  100.0
## Urban          58.0     28.0    15.0  100.0
```

- Here we will talk about the **conditional distribution** of `Goals` given `Urban.Rural`.
- An important question could be:
  - Are the goals of the kids different when they come from urban, suburban or rural areas? I.e. are the rows in the table significantly different?
- There is (almost) no difference between urban and suburban, but it looks like rural is different.

2 Independence

2.1 Independence

- Recall, that two factors are **independent**, when there is no difference between the population's distributions of one factor given the levels of the other factor.
- Otherwise the factors are said to be **dependent**.
- If we e.g. have the following conditional **population distributions** of `Goals` given `Urban.Rural`:

```
##             Goals
## Urban.Rural Grades Popular Sports
##    Rural       500     300    200
##    Suburban    500     300    200
##    Urban       500     300    200
```

- Then the factors `Goals` and `Urban.Rural` are independent.
- We take a sample and "measure" the factors F_1 and F_2. E.g. `Goals` and `Urban.Rural` for a random child.
- The hypothesis of interest today is:

$$H_0 : F_1 \text{ and } F_2 \text{ are independent}, \quad H_a : F_1 \text{ and } F_2 \text{ are dependent}.$$

2.2 The Chi-squared test for independence

- The relative frequencies in the sample gives an estimate of the unconditional distribution of `Goals`:

```
tab = pd.crosstab(popKids['Urban.Rural'], popKids['Goals'])
n = tab.values.sum()
pctGoals = (tab.sum(axis=0) / n * 100).round(1)
pctGoals
```

```
## Goals
## Grades     51.7
## Popular    29.5
## Sports     18.8
## dtype: float64
```

- If we assume independence, then this is also a guess of the conditional distributions of `Goals` given `Urban.Rural`.
- The corresponding expected counts in the sample are then:

```
##            Goals
## Urban.Rural Grades        Popular       Sports        Sum
##    Rural      77.0 (51.7%)  44.0 (29.5%)  28.1 (18.8%) 149.0 (100%)
##    Suburban  78.0 (51.7%)  44.5 (29.5%)  28.4 (18.8%) 151.0 (100%)
##    Urban      92.0 (51.7%)  52.5 (29.5%)  33.5 (18.8%) 178.0 (100%)
##    Sum      247.0 (51.7%) 141.0 (29.5%)  90.0 (18.8%) 478.0 (100%)
```

2.3 Calculation of expected table

```
##            Goals
## Urban.Rural Grades        Popular       Sports        Sum
##    Rural      77.0 (51.7%)  44.0 (29.5%)  28.1 (18.8%) 149.0 (100%)
##    Suburban  78.0 (51.7%)  44.5 (29.5%)  28.4 (18.8%) 151.0 (100%)
##    Urban      92.0 (51.7%)  52.5 (29.5%)  33.5 (18.8%) 178.0 (100%)
##    Sum      247.0 (51.7%) 141.0 (29.5%)  90.0 (18.8%) 478.0 (100%)
```

- We note that
  - The relative frequency for a given column is columnTotal divided by tableTotal. For example `Grades`, which is $\frac{247}{478} = 51.7\%$.
  - The expected value in a given cell in the table is then the cell's relative column frequency multiplied by the cell's rowTotal. For example `Rural` and `Grades`: $149 \times 51.7\% = 77.0$.
- This can be summarized to:
  - The expected value in a cell is the product of the cell's rowTotal and columnTotal divided by tableTotal.

2.4 Chi-squared (χ^2) test statistic

- We have an **observed table**:

```
tab
```

```
## Goals        Grades  Popular  Sports
## Urban.Rural
## Rural            57       50      42
## Suburban         87       42      22
## Urban           103       49      26
```

- And an **expected table**, if H_0 is true:

```
##            Goals
## Urban.Rural Grades Popular Sports Sum
##    Rural      77.0    44.0    28.1  149.0
```

```
##    Suburban  78.0   44.5    28.4  151.0
##    Urban     92.0   52.5    33.5  178.0
##    Sum      247.0  141.0    90.0  478.0
```

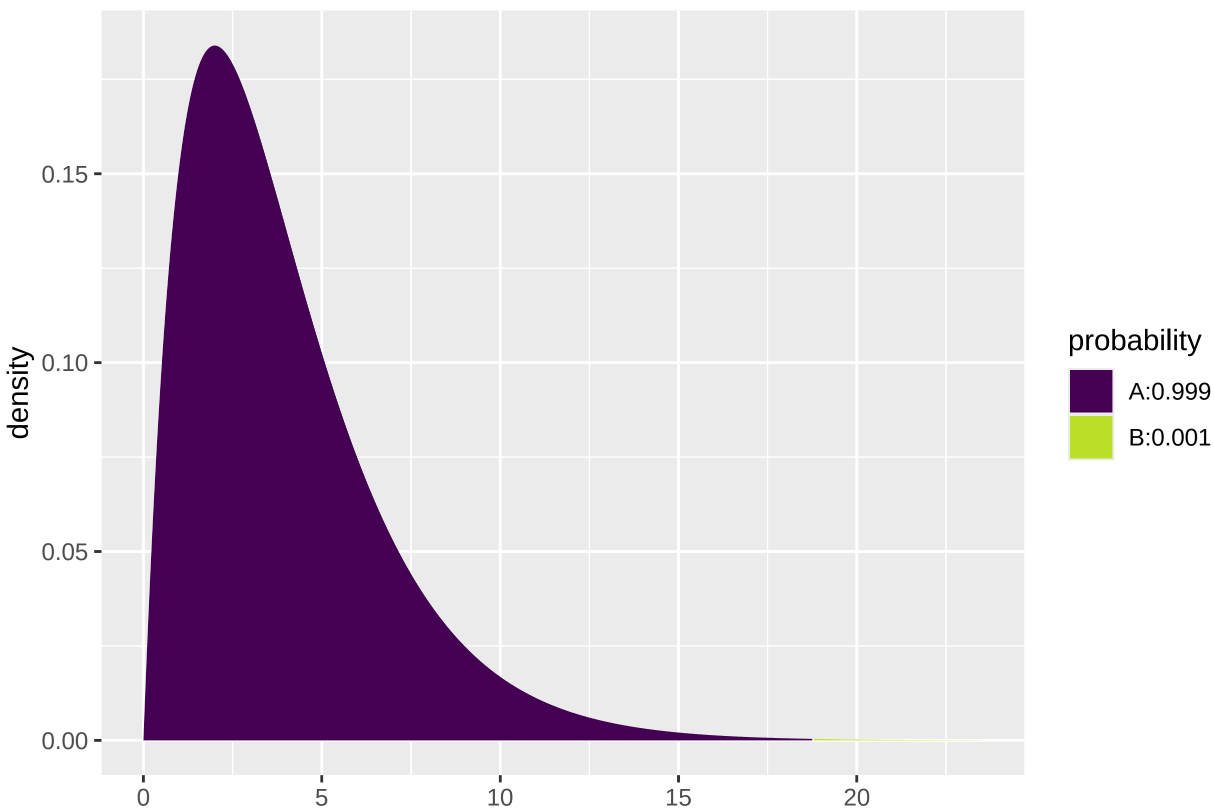
- If these tables are "far from each other", then we reject H_0. We want to measure the distance via the Chi-squared test statistic:
  - $X^2 = \sum \frac{(f_o - f_e)^2}{f_e}$: Sum over all cells in the table
  - f_o is the frequency in a cell in the observed table
  - f_e is the corresponding frequency in the expected table.
- We have:

$$X^2_{obs} = \frac{(57-77)^2}{77} + \ldots + \frac{(26-33.5)^2}{33.5} = 18.8$$

- Is this a large distance??

2.5 χ^2-test template.

- We want to test the hypothesis H_0 of independence in a table with r rows and c columns:
  - We take a sample and calculate X^2_{obs} - the observed value of the test statistic.
  - p-value: Assume H_0 is true. What is then the chance of obtaining a larger X^2 than X^2_{obs}, if we repeat the experiment?
- This can be approximated by the **χ^2-distribution** with $df = (r-1)(c-1)$ degrees of freedom.
- For `Goals` and `Urban.Rural` we have $r = c = 3$, i.e. $df = 4$ and $X^2_{obs} = 18.8$, so the p-value is:




```
from scipy.stats import chi2

1 - chi2.cdf(18.8, 4)
```

```
## np.float64(0.0008603302817890013)
```

- There is clearly a significant association between `Goals` and `Urban.Rural`.

2.6 Perform the test using software

- All of the above calculations can be obtained as follows:

```
import statsmodels.api as sm

tab = pd.crosstab(popKids['Urban.Rural'], popKids['Goals'])
chisqtab = sm.stats.Table(tab)
chisqtab.fittedvalues
```

```
## Goals             Grades    Popular     Sports
## Urban.Rural
## Rural          76.993724  43.951883  28.054393
## Suburban       78.027197  44.541841  28.430962
## Urban          91.979079  52.506276  33.514644
```

```
print(chisqtab.test_nominal_association())
```

```
## df           4
## pvalue       0.0008496551610398528
## statistic    18.827626180696555
```

- The frequency data can also be put directly into a matrix.

```
import numpy as np

data = np.array([
  57, 50, 42,
  87, 42, 22,
  103, 49, 26]).reshape(3,3)
tab = pd.DataFrame(data,
                   index=["Rural", "Suburban", "Urban"],
                   columns=["Grades", "Popular", "Sports"])
tab
```

```
##           Grades  Popular  Sports
## Rural         57       50      42
## Suburban      87       42      22
## Urban        103       49      26
```

```
chisqtab = sm.stats.Table(tab)
chisqtab.fittedvalues
```

```
##              Grades    Popular     Sports
## Rural     76.993724  43.951883  28.054393
## Suburban  78.027197  44.541841  28.430962
## Urban     91.979079  52.506276  33.514644
```

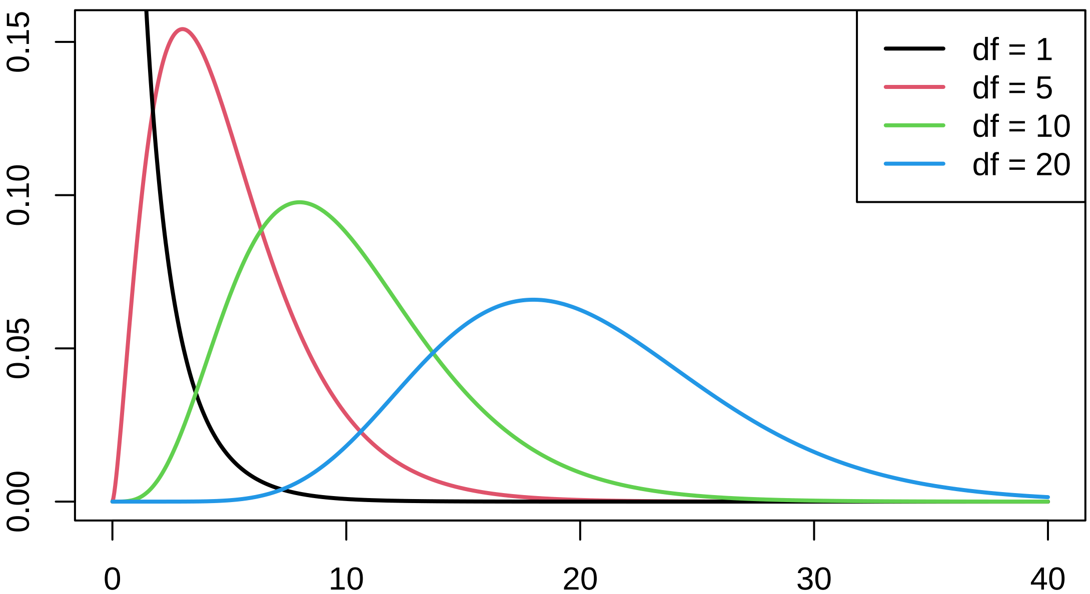
```
print(chisqtab.test_nominal_association())
```

```
## df           4
## pvalue       0.0008496551610398528
## statistic    18.827626180696555
```

3 The χ^2-distribution

3.1 The χ^2-distribution

- The χ^2-distribution with df degrees of freedom:
  - Is never negative.
  - Has mean $\mu = df$
  - Has standard deviation $\sigma = \sqrt{2df}$
  - Is skewed to the right, but approaches a normal distribution when df grows.



4 Agresti - Summary

4.1 Summary

- For the the Chi-squared statistic, X^2, to be appropriate we require that the expected values have to be $f_e \geq 5$.
- Now we can summarize the ingredients in the Chi-squared test for independence.

TABLE 8.5: The Five Parts of the Chi-Squared Test of Independence

1. Assumptions: Two categorical variables, random sampling, $f_e \geq 5$ in all cells
2. Hypotheses: H_0: Statistical independence of variables
   H_a: Statistical dependence of variables
3. Test statistic: $\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e}$, where $f_e = \frac{\text{(Row total)(Column total)}}{\text{Total sample size}}$
4. P-value: P = right-tail probability above observed χ^2 value,
   for chi-squared distribution with $df = (r - 1)(c - 1)$
5. Conclusion: Report P-value
   If decision needed, reject H_0 at α-level if $P \leq \alpha$

5 Standardized residuals

5.1 Residual analysis

- If we reject the hypothesis of independence it can be of interest to identify the significant deviations.
- In a given cell in the table, $f_o - f_e$ is the deviation between data and the expected values under the null hypothesis.
- We assume that $f_e \geq 5$.
- If H_0 is true, then the standard error of $f_o - f_e$ is given by

$$se = \sqrt{f_e(1 - \text{rowProportion})(1 - \text{columnProportion})}$$

- The corresponding z-score

$$z = \frac{f_o - f_e}{se}$$

  should in 95% of the cells be between ± 2. Values above 3 or below -3 should not appear.
- In popKids table cell `Rural and Grade` we got $f_e = 77.0$ and $f_o = 57$. Here columnProportion$= 51.7\%$ and rowProportion$= 149/478 = 31.2\%$.
- We can then calculate

$$z = \frac{57 - 77}{\sqrt{77(1 - 0.517)(1 - 0.312)}} = -3.95$$

  .
- Compared to the null hypothesis there are way too few rural kids who find grades important.
- In summary: The standardized residuals allow for cell-by-cell (f_e vs f_o) comparision.

5.2 Residual analysis

- We can calculate the standardized residuals:

```
import statsmodels.api as sm

tab = pd.crosstab(popKids['Urban.Rural'], popKids['Goals'])
table = sm.stats.Table(tab)
table.standardized_resids
```

```
## Goals          Grades   Popular    Sports
## Urban.Rural
## Rural       -3.950845  1.309623  3.522500
## Suburban     1.766661 -0.548407 -1.618521
## Urban        2.086578 -0.727433 -1.818622
```

5.3 Why not just use two-way ANOVA ?

- number of persons in different categories are *not* normally distributed
- variance typically larger the larger expected frequency
- underlying data are discrete (for each person, which column and row category does person belong to)
- these discrete variables are naturally modelled in terms of probabilies for different categories
- therefore hypothesis of independence becomes natural null hypothesis
- it is possible to model table frequencies as dependent variable using a regression model but then we need the framework of *generalized linear models* (see last slides)

Contingency table:

- *counts* of how many individuals fall within different categories for two (or more) categorical variables

Two-way ANOVA:

- a number of individuals/objects/... available for each combination of two categorical variables
- next a continuous variable is measured for each individual or object (this becomes the response variable)

6 Models for table data

6.1 Example

- We will study the dataset HairEyeColor.

```
HairEyeColor = pd.read_csv("https://asta.math.aau.dk/datasets?file=HairEyeColor.txt", sep="\t")
HairEyeColor.head(6)
```

```
##     Hair    Eye   Sex  Freq
## 0  Black  Brown  Male    32
## 1  Brown  Brown  Male    53
## 2    Red  Brown  Male    10
## 3  Blond  Brown  Male     3
## 4  Black   Blue  Male    11
## 5  Brown   Blue  Male    50
```

- Data is organized such that the variable Freq gives the frequency of each combination of the factors Hair, Eye and Sex.
- For example: 32 observations are men with black hair and brown eyes.
- We are interested in the association between eye color and hair color ignoring the sex
- We aggregate data, so we have a table with frequencies for each combination of Hair and Eye.

```
HairEye = HairEyeColor.groupby(['Eye', 'Hair'], as_index=False)['Freq'].sum()
HairEye
```

```
##       Eye   Hair  Freq
## 0    Blue  Black    20
## 1    Blue  Blond    94
## 2    Blue  Brown    84
## 3    Blue    Red    17
## 4   Brown  Black    68
## 5   Brown  Blond     7
## 6   Brown  Brown   119
## 7   Brown    Red    26
## 8   Green  Black     5
## 9   Green  Blond    16
## 10  Green  Brown    29
## 11  Green    Red    14
## 12  Hazel  Black    15
## 13  Hazel  Blond    10
## 14  Hazel  Brown    54
## 15  Hazel    Red    14
```

6.2 Model specification

- We can write down a model for (the logarithm of) the expected frequencies by using dummy variables z_{e1}, z_{e2}, z_{e3} and z_{h1}, z_{h2}, z_{h3}
- To denote the different levels of Eye and Hair (the reference level has all dummy variables equal to 0):

$$\log(f_e) = \alpha + \beta_{e1}z_{e1} + \beta_{e2}z_{e2} + \beta_{e3}z_{e3} + \beta_{h1}z_{h1} + \beta_{h2}z_{h2} + \beta_{h3}z_{h3}.$$

- Note that we haven't included an interaction term, which is this case implies, that we assume independence between Eye and Hair in the model.
- Since our response variable now is a count it is no longer a linear model as we have been used to (linear regression).
- Instead it is a so-called generalized linear model and the relevant command is glm.

6.3 Model specification

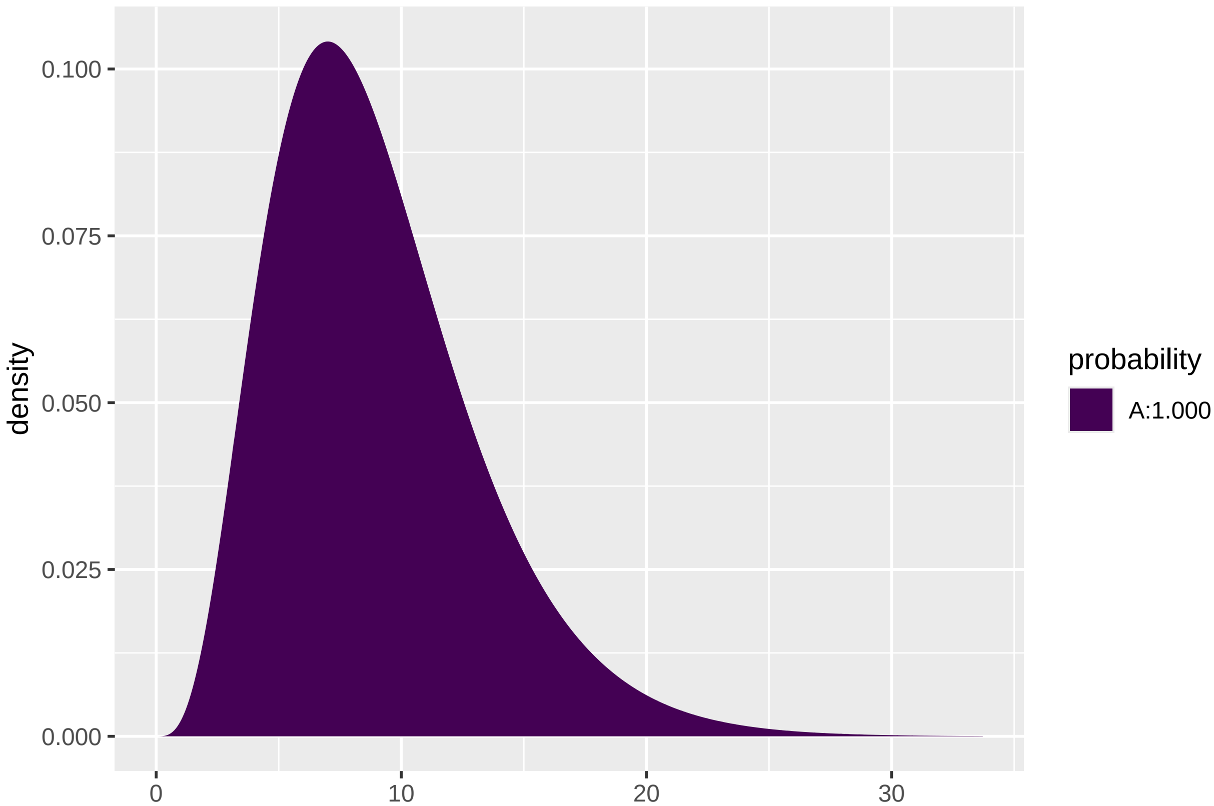
```
import statsmodels.formula.api as smf

model = smf.glm('Freq ~ Hair + Eye', data=HairEye, family=sm.families.Poisson()).fit()
```

- The `family` argument (Poisson) ensures that data are interpreted as discrete counts and not a continuous variable.

```
model.summary()
```

```
## <class 'statsmodels.iolib.summary.Summary'>
## """
##                  Generalized Linear Model Regression Results
## ==============================================================================
## Dep. Variable:                   Freq   No. Observations:                   16
## Model:                            GLM   Df Residuals:                        9
## Model Family:                 Poisson   Df Model:                            6
## Link Function:                    Log   Scale:                          1.0000
## Method:                          IRLS   Log-Likelihood:                -113.52
## Date:                Mon, 10 Aug 2026   Deviance:                       146.44
## Time:                        15:36:01   Pearson chi2:                     138.
## No. Iterations:                     5   Pseudo R-squ. (CS):              1.000
## Covariance Type:            nonrobust
## =================================================================================
##                     coef    std err          z      P>|z|      [0.025      0.975]
## ---------------------------------------------------------------------------------
## Intercept         3.6693      0.111     33.191      0.000       3.453       3.886
## Hair[T.Blond]     0.1621      0.131      1.238      0.216      -0.094       0.419
## Hair[T.Brown]     0.9739      0.113      8.623      0.000       0.752       1.195
## Hair[T.Red]      -0.4195      0.153     -2.745      0.006      -0.719      -0.120
## Eye[T.Brown]      0.0230      0.096      0.240      0.811      -0.165       0.211
## Eye[T.Green]     -1.2118      0.142     -8.510      0.000      -1.491      -0.933
## Eye[T.Hazel]     -0.8380      0.124     -6.752      0.000      -1.081      -0.595
## =================================================================================
## """
```

- A deviance value of $X^2 = 146.44$ with $df = 9$ shows that there is very clear significance and we reject the null hypothesis of independence between hair and eye color.




```
1 - chi2.cdf(146.44, 9)
```

```
## np.float64(0.0)
```

6.4 Expected values and standardized residuals

- We also want to look at expected values and standardized (studentized) residuals.
- The null hypothesis predicts $e^{3.67+0.02} = 40.1$ with brown eyes and black hair, but we have observed 68.
- This is significantly too many, since the standardized residual is 6.1.
- The null hypothesis predicts 47.2 with brown eyes and blond hair, but we have seen 7. This is significantly too few, since the standardized residual is -8.3.

```
HairEye['fitted'] = model.fittedvalues
HairEye['resid'] = model.get_influence().resid_studentized
HairEye
```

```
##       Eye   Hair  Freq      fitted     resid
## 0    Blue  Black    20   39.222973 -4.253816
## 1    Blue  Blond    94   46.123311  9.967550
## 2    Blue  Brown    84  103.868243 -3.397883
## 3    Blue    Red    17   25.785473 -2.311052
## 4   Brown  Black    68   40.135135  6.136520
## 5   Brown  Blond     7   47.195946 -8.328248
## 6   Brown  Brown   119  106.283784  2.164282
## 7   Brown    Red    26   26.385135 -0.100824
## 8   Green  Black     5   11.675676 -2.287896
## 9   Green  Blond    16   13.729730  0.732023
## 10  Green  Brown    29   30.918919 -0.508263
```

```
## 11  Green    Red    14    7.675676  2.576569
## 12  Hazel  Black    15   16.966216 -0.575026
## 13  Hazel  Blond    10   19.951014 -2.737977
## 14  Hazel  Brown    54   44.929054  2.050216
## 15  Hazel    Red    14   11.153716  0.989512
```