Comparison of two groups

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Contents

0.1 Response variable and explanatory variable

- We conduct an experiment, where we at random choose 50 IT-companies and 50 service companies and measure their profit ratio. Is there association between company type (IT/service) and profit ratio?
- In other words we compare samples from 2 different populations. For each company we register:
  - The binary variable `company type`, which is called **the explanatory variable** and divides data in 2 groups.
  - The quantitative variable `profit ratio`, which is called **the response variable**.

0.2 Dependent/independent samples

- In the example with profit ratio of 50 IT-companies and 50 service companies we have **independent samples**, since the same company cannot be in both groups.
- Now, think of another type of experiment, where we at random choose 50 IT-companies and measure their profit ratio in both 2009 and 2010. Then we may be interested in whether there is association between year and profit ratio?
- In this example we have **dependent samples**, since the same company is in both groups.
- Dependent samples may also be referred to as paired samples.

0.3 Comparison of two means (Independent samples)

- We consider the situation, where we have two quantitative samples:
  - Population 1 has mean μ_1, which is estimated by $\hat{\mu}_1 = \bar{y}_1$ based on a sample of size n_1.

  - Population 2 has mean μ_2, which is estimated by $\hat{\mu}_2 = \bar{y}_2$ based on a sample of size n_2.
  - We are interested in the difference $\mu_2 - \mu_1$, which is estimated by $d = \bar{y}_2 - \bar{y}_1$.
  - Assume that we can find the **estimated standard error** se_d of the difference and that this has degrees of freedom df.
  - Assume that the samples either are large or come from a normal population.
- Then we can construct a
  - confidence interval for the unknown population difference of means $\mu_2 - \mu_1$ by

$$(\bar{y}_2 - \bar{y}_1) \pm t_{crit} se_d,$$

    where the critical t-score, t_{crit}, determines the confidence level.
  - significance test:
    * for the null hypothesis $H_0: \ \mu_2 - \mu_1 = 0$ and alternative hypothesis $H_a: \ \mu_2 - \mu_1 \neq 0$.
    * which uses the test statistic: $t_{obs} = \frac{(\bar{y}_2 - \bar{y}_1) - 0}{se_d}$, that has to be evaluated in a t-distribution with df degrees of freedom.

0.4 Comparison of two means (Independent samples)

- In the independent samples situation it can be shown that

$$se_d = \sqrt{se_1^2 + se_2^2},$$

  where se_1 and se_2 are estimated standard errors for the sample means in populations 1 and 2, respectively.
- We recall, that for these we have $se = \frac{s}{\sqrt{n}}$, i.e.

$$se_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},$$

  where s_1 and s_2 are estimated standard deviations for population 1 and 2, respectively.
- **The degrees of freedom** df for se_d can be estimated by a complicated formula, which we will not present here.
- For the confidence interval and the significance test we note that:
  - If both n_1 and n_2 are above 30, then we can use the standard normal distribution (z-score) rather than the t-distribution (t-score).
  - If n_1 or n_2 are below 30, then we let **R** calculate the degrees of freedom and p-value/confidence interval.

0.5 Example: Comparing two means (independent samples)

We return to the `Chile` data. We study the association between the variables `sex` and `statusquo` (scale of support for the status-quo). So, we will perform a significance test to test for difference in the mean of `statusquo` for male and females.

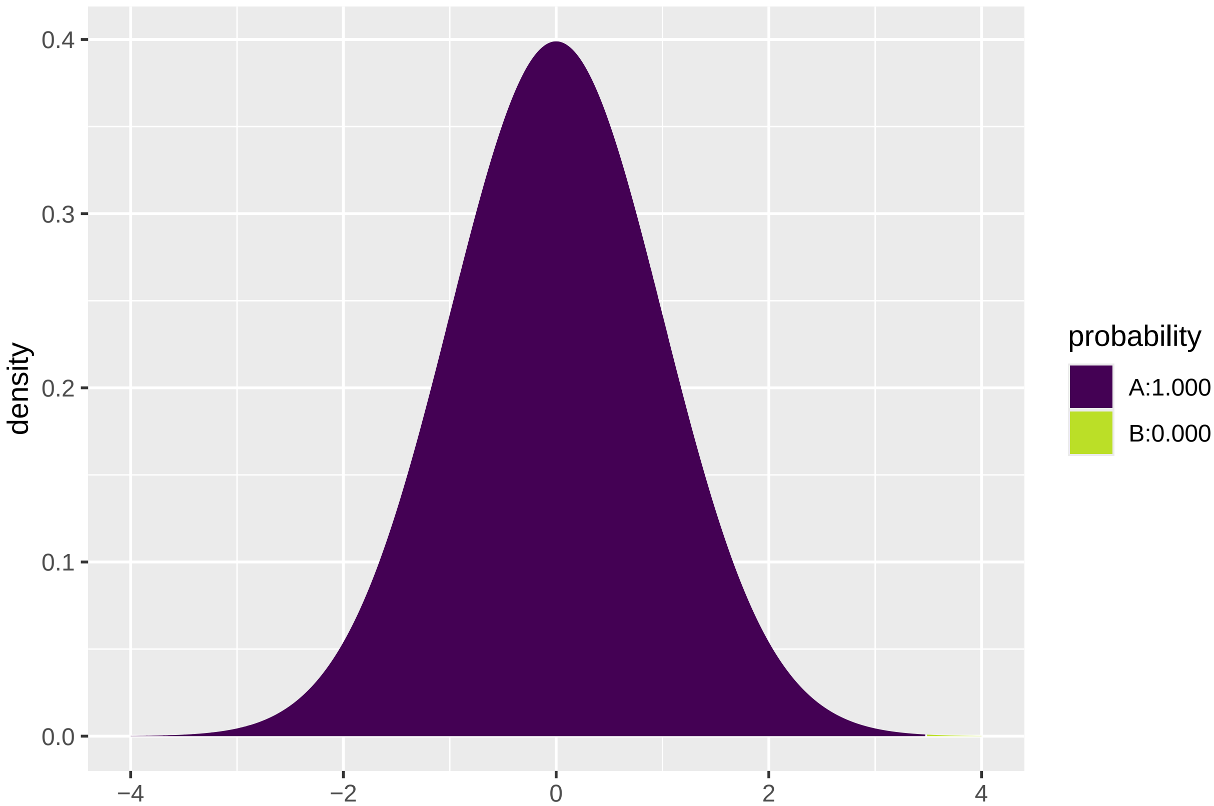
```
Chile <- read.delim("https://asta.math.aau.dk/datasets?file=Chile.txt")
library(mosaic)
fv <- favstats(statusquo ~ sex, data = Chile)
fv
```

```
##   sex   min     Q1 median    Q3  max    mean    sd    n missing
## 1   F -1.80 -0.975  0.121 1.033 2.02  0.0657 1.003 1368      11
## 2   M -1.74 -1.032 -0.216 0.861 2.05 -0.0684 0.993 1315       6
```

- Difference: $d = 0.0657 - (-0.0684) = 0.1341$.
- Estimated standard deviations: $s_1 = 1.0032$ (females) and $s_2 = 0.9928$ (males).
- Sample sizes: $n_1 = 1368$ and $n_2 = 1315$.
- Estimated standard error of difference: $se_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{1.0032^2}{1368} + \frac{0.9928^2}{1315}} = 0.0385$.

- Observed t-score for $H_0: \ \mu_1 - \mu_2 = 0$ is: $\quad t_{obs} = \frac{d-0}{se_d} = \frac{0.1341}{0.0385} = 3.4786$.
- Since both sample sizes are "pretty large" (> 30), we can use the z-score instead of the t-score for finding the p-value (i.e. we use the standard normal distribution):

```
1 - pdist("norm", q = 3.4786, xlim = c(-4, 4))
```




```
## [1] 0.0002520202
```

- Then the p-value is $2 \cdot 0.00025 = 0.0005$, so we reject the null hypothesis.
- We can leave all the calculations to **R** by using `t.test`:

```
t.test(statusquo ~ sex, data = Chile)
```

```
##
##  Welch Two Sample t-test
##
## data:  statusquo by sex
## t = 3.4786, df = 2678.7, p-value = 0.0005121
## alternative hypothesis: true difference in means between group F and group M is not equal to 0
## 95 percent confidence interval:
##  0.05849179 0.20962982
## sample estimates:
## mean in group F mean in group M
##      0.06570627     -0.06835453
```

- We recognize the t-score 3.4786 and the p-value 0.0005. The estimated degrees of freedom $df = 2679$ is so large that we can not tell the difference between results obtained using z-score and t-score.

0.6 Comparison of two means: confidence interval (independent samples)

- We have already found all the ingredients to construct a **confidence interval for** $\mu_2 - \mu_1$:

- $d = \bar{y}_2 - \bar{y}_1$ estimates $\mu_2 - \mu_1$.
- $se_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$ estimates the standard error of d.

- Then:
$$d \pm t_{crit} se_d$$
is a confidence interval for $\mu_2 - \mu_1$.
- The critical t-score, t_{crit} is chosen corresponding to the wanted confidence level. If n_1 and n_2 both are greater than 30, then $t_{crit} = 2$ yields a confidence level of approximately 95%.

0.7 Comparison of two means: paired t-test (dependent samples)

- Experiment:
  - You choose 32 students at random and measure their average reaction time in a driving simulator while they are listening to radio or audio books.
  - Later the same 32 students redo the simulated driving while talking on a cell phone.
- It is interesting to investigate whether or not the fact that you are actively participating in a conversation changes your average reaction time compared to when you are passively listening.
- So we have 2 samples corresponding to with/without phone. In this case we have **dependent** samples, since we have 2 measurement for each student.
- We use the following strategy for analysis:
  - For each student calculate **the change** in average reaction time with and without talking on the phone.
  - The changes $d_1, d_2, \ldots, d_{32}$ are now considered as **ONE** sample from a population with mean μ.
  - Test the hypothesis $H_0 : \mu = 0$ as usual (using a t-test for testing the mean as in the previous lecture).

0.7.1 Reaction time example

- Data is organized in a data frame with 3 variables:
  - `student` (integer – a simple id)
  - `reaction_time` (numeric – average reaction time in milliseconds)
  - `phone` (factor – `yes`/`no` indicating whether speaking on the phone)

```
reaction <- read.delim("https://asta.math.aau.dk/datasets?file=reaction.txt")
head(reaction, n = 3)
```

```
##   student reaction_time phone
## 1       1           604    no
## 2       2           556    no
## 3       3           540    no
```

Instead of doing manual calculations we let **R** perform the significance test (using `t.test` with `paired = TRUE` as our samples are paired/dependent):

```
yes <- subset(reaction, phone == "yes")
no  <- subset(reaction, phone == "no")
all(yes$student == no$student)
```

```
## [1] TRUE
```

```
reaction_paired <- data.frame(student = no$student, yes = yes$reaction_time, no = no$reaction_time)
t.test(reaction_paired$no, reaction_paired$yes, paired = TRUE)
```

```
##
##  Paired t-test
##
```

```
## data:  reaction_paired$no and reaction_paired$yes
## t = -5.4563, df = 31, p-value = 5.803e-06
## alternative hypothesis: true mean difference is not equal to 0
## 95 percent confidence interval:
##  -69.54814 -31.70186
## sample estimates:
## mean difference
##         -50.625
```

- With a p-value of 0.0000058 we reject that speaking on the phone has no influence on the reaction time.
- To understand what is going on, we can manually find the reaction time difference for each student and do a one sample t-test on this difference:

```
reaction_paired$diff <- reaction_paired$yes - reaction_paired$no
head(reaction_paired)
```

```
##   student yes  no diff
## 1       1 636 604   32
## 2       2 623 556   67
## 3       3 615 540   75
## 4       4 672 522  150
## 5       5 601 459  142
## 6       6 600 544   56
```

```
t.test( ~ diff, data = reaction_paired)
```

```
##
##  One Sample t-test
##
## data:  diff
## t = 5.4563, df = 31, p-value = 5.803e-06
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  31.70186 69.54814
## sample estimates:
## mean of x
##    50.625
```

1 Comparison of two proportions

1.1 Comparison of two proportions

- We consider the situation, where we have two qualitative samples and we investigate whether a given property is present or not:
  - Let the proportion of population 1 which has the property be π_1, which is estimated by $\hat{\pi}_1$ based on a sample of size n_1.
  - Let the proportion of population 2 which has the property be π_2, which is estimated by $\hat{\pi}_2$ based on a sample of size n_2.
  - We are interested in the difference $\pi_2 - \pi_1$, which is estimated by $d = \hat{\pi}_2 - \hat{\pi}_1$.
  - Assume that we can find the **estimated standard error** se_d of the difference.
- Then we can construct
  - an approximate confidence interval for the difference, $\pi_2 - \pi_1$.
  - a significance test.

1.2 Comparison of two proportions: Independent samples

- In the situation where we have independent samples we know that

$$se_d = \sqrt{se_1^2 + se_2^2},$$

  where se_1 and se_2 are the estimated standard errors for the sample proportion in population 1 and 2, respectively.
- We recall, that these are given by $se = \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$, i.e.

$$se_d = \sqrt{\frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_2}}.$$

- A (approximate) confidence interval for $\pi_2 - \pi_1$ is obtained by the usual construction:

$$(\hat{\pi}_2 - \hat{\pi}_1) \pm z_{crit} se_d,$$

  where the critical z-score determines the confidence level.

1.3 Approximate test for comparing two proportions (independent samples)

- We consider the null hypothesis H_0: $\pi_1 = \pi_2$ (equivalently $H_0 : \pi_1 - \pi_2 = 0$) and the alternative hypothesis H_a: $\pi_1 \neq \pi_2$.
- Assuming H_0 is true, we have a common proportion π, which is estimated by

$$\hat{\pi} = \frac{n_1\hat{\pi}_1 + n_2\hat{\pi}_2}{n_1 + n_2},$$

  i.e. we aggregate the populations and calculate the relative frequency of the property (with other words: we estimate the proportion, π, as if the two samples were one).
- Rather than using the estimated standard error of the difference from previous, we use the following that holds under H_0:

$$se_0 = \sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

- The observed test statistic/z-score for H_0 is then:

$$z_{obs} = \frac{(\hat{\pi}_2 - \hat{\pi}_1) - 0}{se_0},$$

  which is evaluated in the standard normal distribution.
- The p-value is calculated in the usual way.

WARNING: The approximation is only good, when $n_1\hat{\pi},\ n_1(1-\hat{\pi}),\ n_2\hat{\pi},\ n_2(1-\hat{\pi})$ all are greater than 5.

1.4 Example: Approximate confidence interval and test for comparing proportions

We return to the `Chile` dataset. We make a new binary variable indicating whether the person intends to vote no or something else (and we remember to tell **R** that it should think of this as a grouping variable, i.e. a `factor`):

```
Chile$voteNo <- relevel(factor(Chile$vote == "N"), ref = "TRUE")
```

We study the association between the variables `sex` and `voteNo`:

```r
tab <- tally( ~ sex + voteNo, data = Chile, useNA = "no")
tab
```

```
##    voteNo
## sex TRUE FALSE
##   F  363   946
##   M  526   697
```

This gives us all the ingredients needed in the hypothesis test:

- Estimated proportion of men that vote no: $\hat{\pi}_1 = \frac{526}{526+697} = 0.430$
- Estimated proportion of women that vote no: $\hat{\pi}_2 = \frac{363}{363+946} = 0.277$

1.5 Example: Approximate confidence interval (cont.)

- Estimated difference:

$$d = \hat{\pi}_2 - \hat{\pi}_1 = 0.277 - 0.430 = -0.153$$

- Standard error of difference:

$$\begin{aligned} se_d &= \sqrt{\frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_2}} \\ &= \sqrt{\frac{0.430(1-0.430)}{1223} + \frac{0.277(1-0.277)}{1309}} = 0.0188. \end{aligned}$$

- Approximate 95% confidence interval for difference:

$$d \pm 1.96 \cdot se_d = (-0.190, -0.116).$$

1.6 Example: p-value (cont.)

- Estimated common proportion:

$$\hat{\pi} = \frac{1223 \times 0.430 + 1309 \times 0.277}{1309 + 1223} = \frac{526 + 363}{1309 + 1223} = 0.351.$$

- Standard error of difference when $H_0: \ \pi_1 = \pi_2$ is true:

$$se_0 = \sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = 0.0190.$$

- The observed test statistic/z-score:

$$z_{obs} = \frac{d}{se_0} = -8.06.$$

- The test for H_0 against $H_a: \pi_1 \neq \pi_2$ yields a p-value that is practically zero, i.e. we can reject that the proportions are equal.

1.7 Automatic calculation in R

```r
Chile2 <- subset(Chile, !is.na(voteNo))
prop.test(voteNo ~ sex, data = Chile2, correct = FALSE)
```

```
##
##  2-sample test for equality of proportions without continuity correction
##
```

```
## data:  tally(voteNo ~ sex)
## X-squared = 64.777, df = 1, p-value = 8.389e-16
## alternative hypothesis: two.sided
## 95 percent confidence interval:
##  -0.1896305 -0.1159275
## sample estimates:
##    prop 1    prop 2
## 0.2773109 0.4300899
```

1.8 Fisher's exact test

- If $n_1\hat{\pi},\ n_1(1-\hat{\pi}),\ n_2\hat{\pi},\ n_2(1-\hat{\pi})$ are not all greater than 5, then the approximate test cannot be trusted. Instead you can use Fisher's exact test:

```
fisher.test(tab)
```

```
##
##  Fisher's Exact Test for Count Data
##
## data:  tab
## p-value = 1.04e-15
## alternative hypothesis: true odds ratio is not equal to 1
## 95 percent confidence interval:
##  0.4292768 0.6021525
## sample estimates:
## odds ratio
##  0.5085996
```

- Again the p-value is seen to be extremely small, so we definitely reject the null hypothesis of equal `voteNo` proportions for women and men.

1.9 Agresti: Overview of comparison of two groups

TABLE 7.10: Summary of Comparison Methods for Two Groups, for Independent Random Samples

	Type of Response Variable	
	Categorical	Quantitative
Estimation		
1. Parameter	$\pi_2 - \pi_1$	$\mu_2 - \mu_1$
2. Point estimate	$\hat{\pi}_2 - \hat{\pi}_1$	$\bar{y}_2 - \bar{y}_1$
3. Standard error	$se = \sqrt{\frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_2}}$	$se = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$
4. Confidence interval	$(\hat{\pi}_2 - \hat{\pi}_1) \pm z(se)$	$(\bar{y}_2 - \bar{y}_1) \pm t(se)$
Significance testing		
1. Assumptions	Randomization $\geq$10 observations in each category, for each group	Randomization Normal population dist.'s (robust, especially for large *n*'s)
2. Hypotheses	H_0: $\pi_1 = \pi_2$ $(\pi_2 - \pi_1 = 0)$ H_a: $\pi_1 \neq \pi_2$	H_0: $\mu_1 = \mu_2$ $(\mu_2 - \mu_1 = 0)$ H_a: $\mu_1 \neq \mu_2$
3. Test statistic	$z = \frac{\hat{\pi}_2 - \hat{\pi}_1}{se_0}$	$t = \frac{\bar{y}_2 - \bar{y}_1}{se}$
4. *P*-value	Two-tail probability from standard normal or *t* (Use one tail for one-sided alternative)	