Stochastic processes III

The ASTA team

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1 Models with exogenous variables

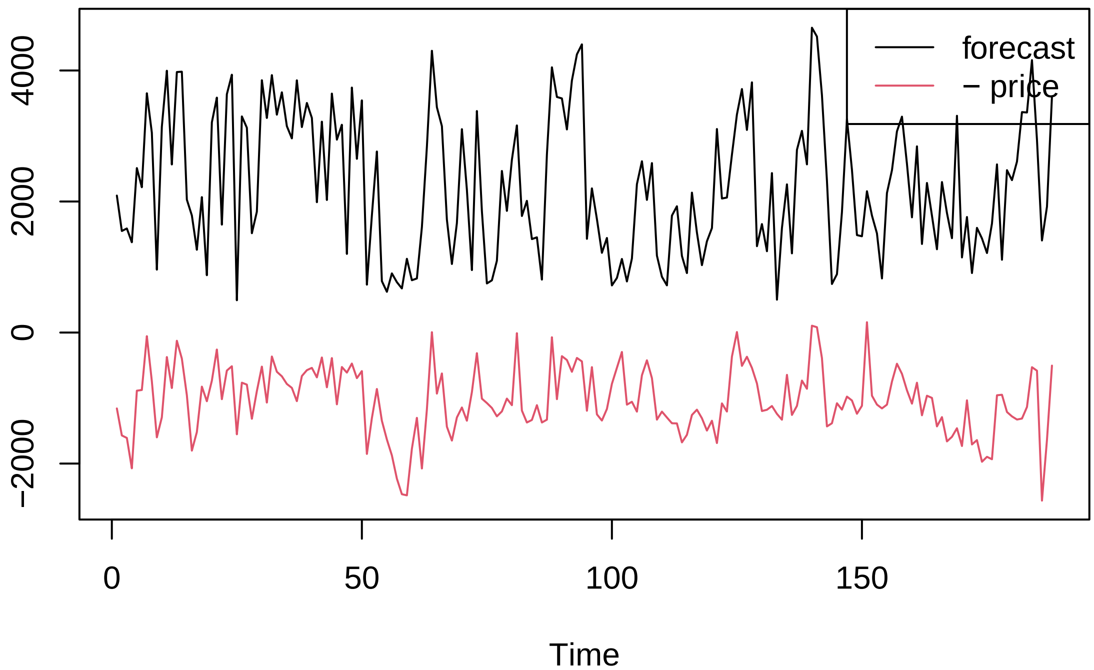
1.1 Exogenous variables

- The ARMA processes are flexible models for a time series Y_t, $t = 1, \dots, n$ evolving randomly over time, but they do not include the possibility that anything is influencing Y_t.
- An exogeneous variable is another variable, say X_t, that influences the behaviour of Y_t
  - Wind power production Y_t is influenced by the wind speed X_t
  - The velocity of a DC motor Y_t is influenced by the input voltage X_t
- Here X_t may be another stochastic process, which we do not model, but only consider as given, or it might be something we can control.

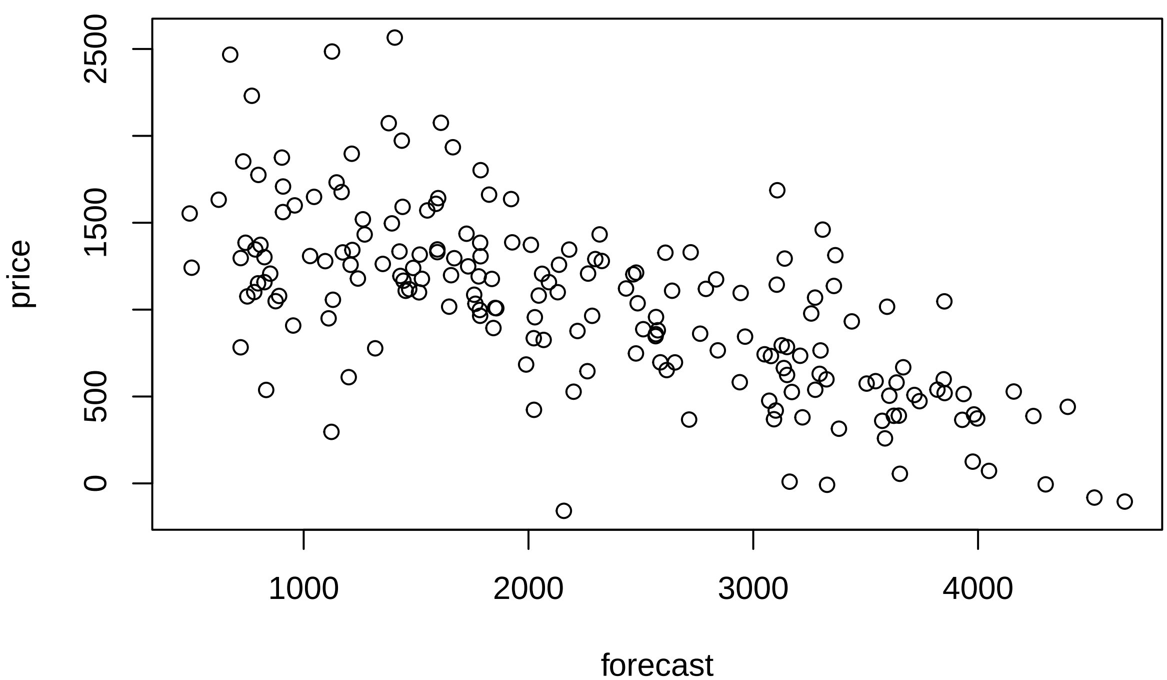
1.2 Data example

- The dataset below contains data from Jan 7 to Jul 13 2022 on two variables
  - forecast: Total day ahead forecasted wind and solar energy production
  - price: Day ahead elspot prices with weekly variation removed

```r
elspot<-read.csv("https://asta.math.aau.dk/eng/static/datasets?file=elspot.csv", header = TRUE)
forecast<-elspot[,2]
price<-elspot[,3]
ts.plot(ts(forecast),ts(-price),col=1:2)
legend("topright",legend=c("forecast","- price"),col=1:2,lty=1)
```




```r
plot(forecast,price)
```



1.3 Regression models with exogenous variables

- We can combine regression models with ARMA models to obtain a stochastic process which is influenced by exogenous variables.
- Consider a linear regression of Y_t on X_t, but where the noise term is an ARMA process:

$$Y_t = \gamma_0 + \gamma_1 X_t + \epsilon_t, \qquad \alpha(B)\epsilon_t = \beta(B)W_t$$

- If we isolate $\epsilon_t = Y_t - \gamma_0 - \gamma_1 X_t$ and insert into the ARMA expression, we get something that looks more like an ARMA process, but with Y_t adjusted by the exogenous variable:

$$\alpha(B)(Y_t - \gamma_0 - \gamma_1 X_t) = \beta(B)W_t$$

- The purpose of fitting such a model is both to obtain a good model for the evolution of the data and to obtain an understanding of the relation between Y_t and X_t.
- Above, X_t is a single stochastic process, but we can also include multiple stochastic processes by making a multiple regression model with an ARMA model for the errors.

1.4 Example

- As an example consider a simple linear regression combined with an AR(1) process for noise terms:

$$Y_t = \gamma_0 + \gamma_1 X_t + \epsilon_t, \qquad \epsilon_t = \alpha_1 \epsilon_{t-1} + W_t$$

  or, since $\epsilon_{t-1} = Y_{t-1} - \gamma_0 - \gamma_1 X_{t-1}$,

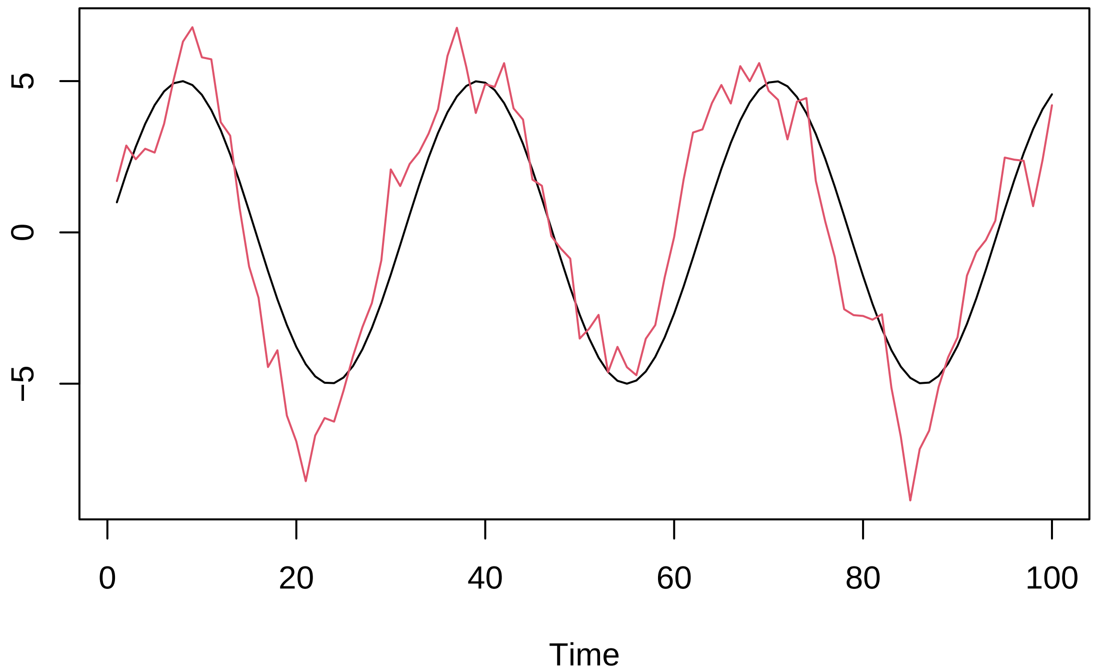
$$Y_t = \alpha_1 Y_{t-1} + (1 - \alpha_1)\gamma_0 + \gamma_1 (X_t - \alpha_1 X_{t-1}) + W_t$$

- Notice that the model behaves like an AR(1) process, but instead of having a constant mean of 0, its mean is constantly adjusted by the exogenous variable.

1.5 Simulation of the example

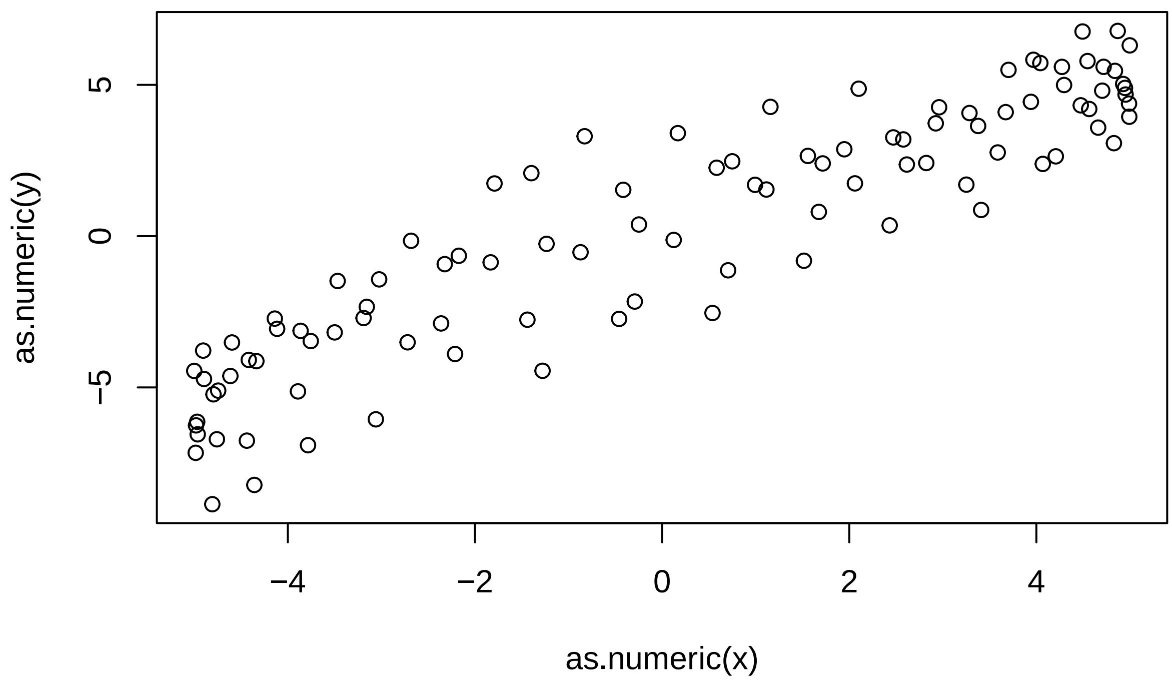
- We simulate some data resembling the example, where we let X_t follow a sine curve:

```
alpha = 0.9; gamma = 1; n = 100
x = as.ts(5*sin(1:n/5))
eps = arima.sim(model=list(ar=alpha),n=n)
y = gamma*x+eps
ts.plot(x,y,col=1:2)
```



- We should think of the red curve as some data we want to model, and the black curve as another variable which we believe may influence the data.
- We can also plot X_t against Y_t to get a view of the relation between the two variables.

```
plot(as.numeric(x),as.numeric(y))
```



1.6 Estimation and model checking

- We can estimate the parameters using the arima function in R.

- We fit a linear regression model with AR(1) noise to the simulated data (i.e. the true model used for simulation):

```
mod=arima(y,order=c(1,0,0),xreg=x); mod
```

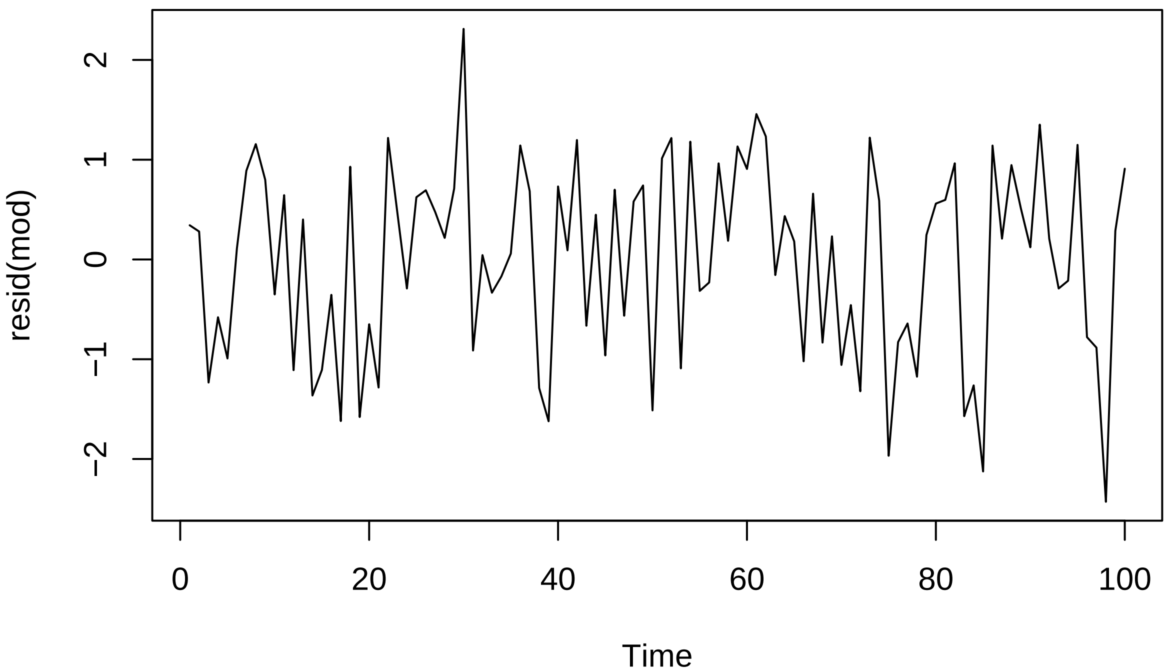
```
##
## Call:
## arima(x = y, order = c(1, 0, 0), xreg = x)
##
## Coefficients:
##          ar1  intercept       x
##       0.8069     0.0679  1.0551
## s.e.  0.0569     0.4795  0.1018
##
## sigma^2 estimated as 0.923:  log likelihood = -138.41,  aic = 284.82
```

- The fitted model becomes

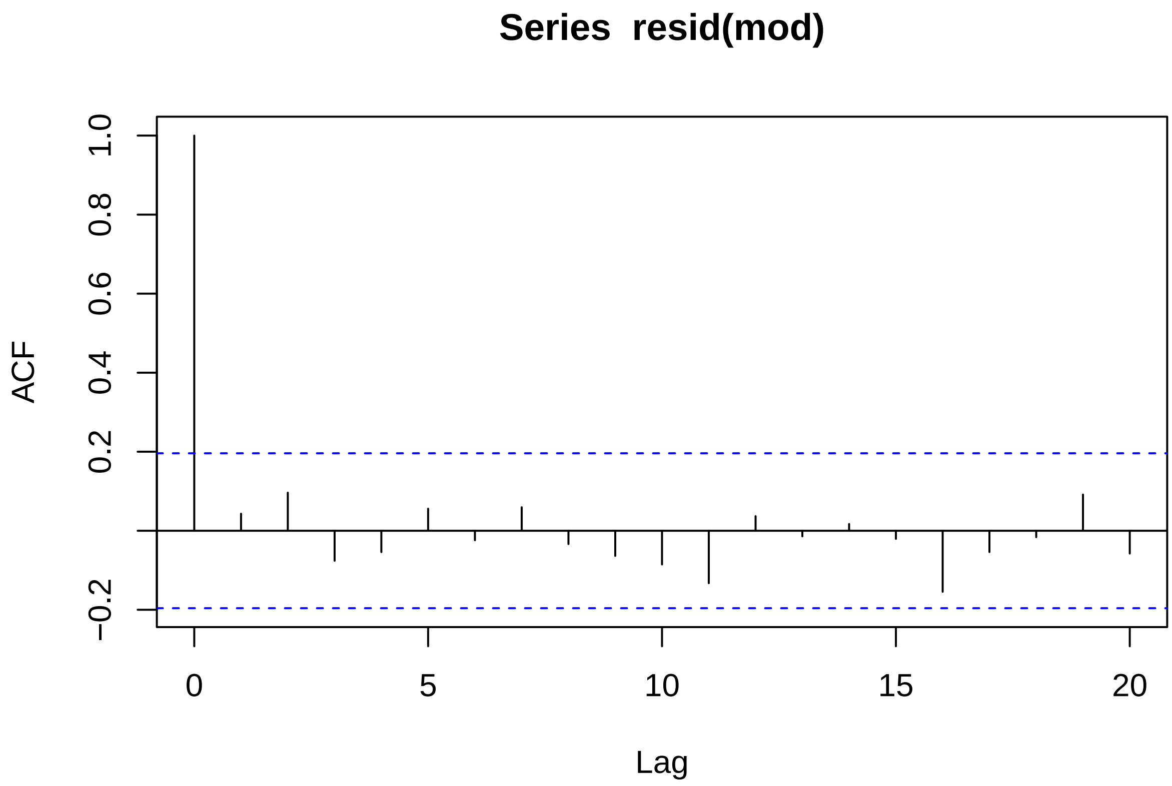
$$Y_t = 0.0679 + 1.0551 \cdot X_t + \epsilon_t, \qquad \epsilon_t = 0.8069 \cdot \epsilon_{t-1} + W_t$$

- The errors $\hat{\epsilon}_t = y_t - 0.0679 + 1.0551 \cdot x_t$ should behave like an AR(1)-model with $\hat{\alpha} = 0.8069$.
    - So the residuals $\epsilon_t - 0.8069 \cdot \epsilon_{t-1}$ should look like white noise.

```
plot(resid(mod))
```




```
acf(resid(mod))
```



1.7 Fitting AR(1) model to data example

- Recall the elspot price dataset

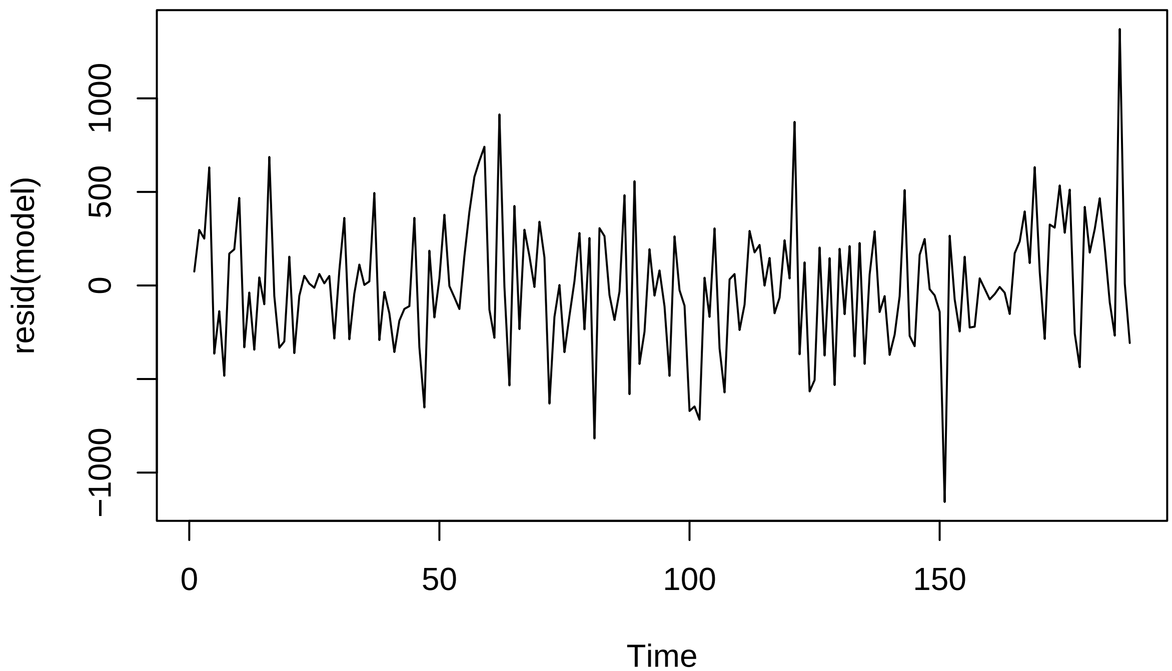
```
forecast<- ts(forecast)
price<-ts(price)
model=arima(price,order=c(1,0,0),xreg=forecast); model
```

```
##
## Call:
## arima(x = price, order = c(1, 0, 0), xreg = forecast)
##
## Coefficients:
##          ar1  intercept  forecast
##       0.3886  1715.8412   -0.3053
## s.e.  0.0680    73.2894    0.0271
##
## sigma^2 estimated as 117486:  log likelihood = -1364.2,  aic = 2736.41
```

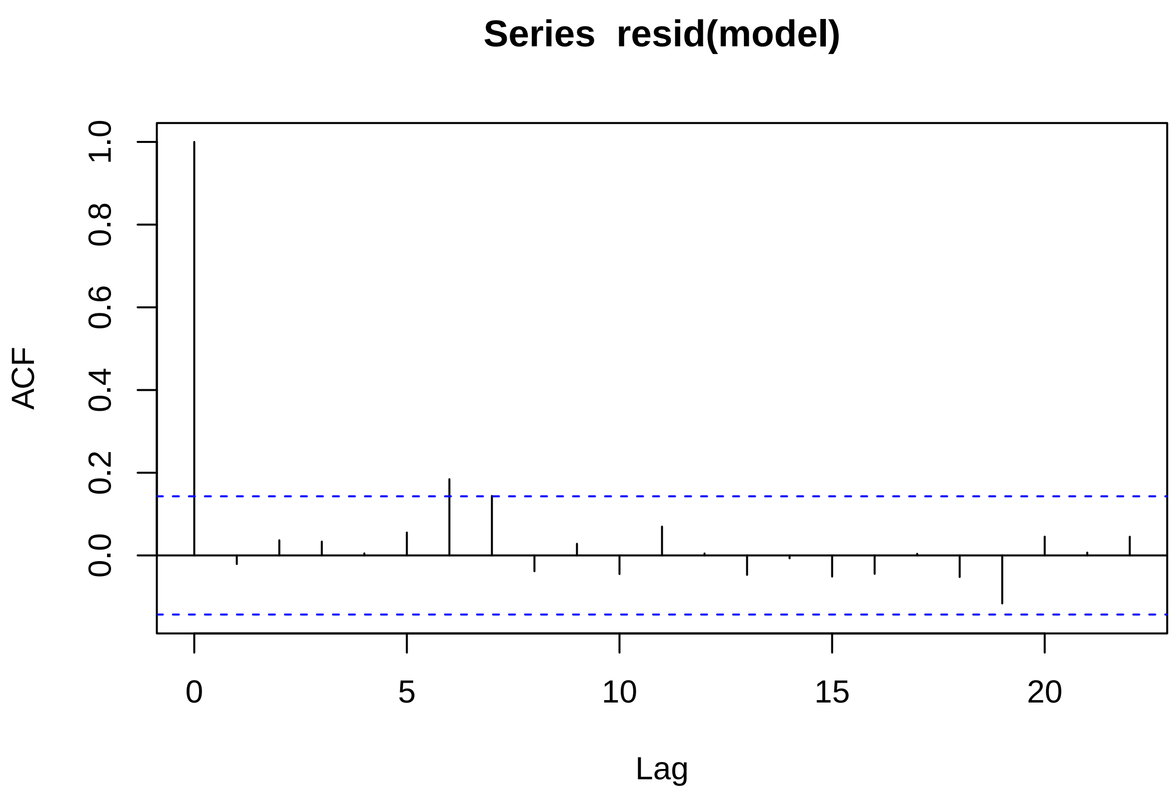
- So we get the model

$$\text{price}_t = 1715.8412 - 0.3053 \cdot \text{forecast}_t + \epsilon_t, \qquad \epsilon_t = 0.3886 \cdot \epsilon_{t-1} + W_t.$$

```r
plot(resid(model))
```




```r
acf(resid(model))
```

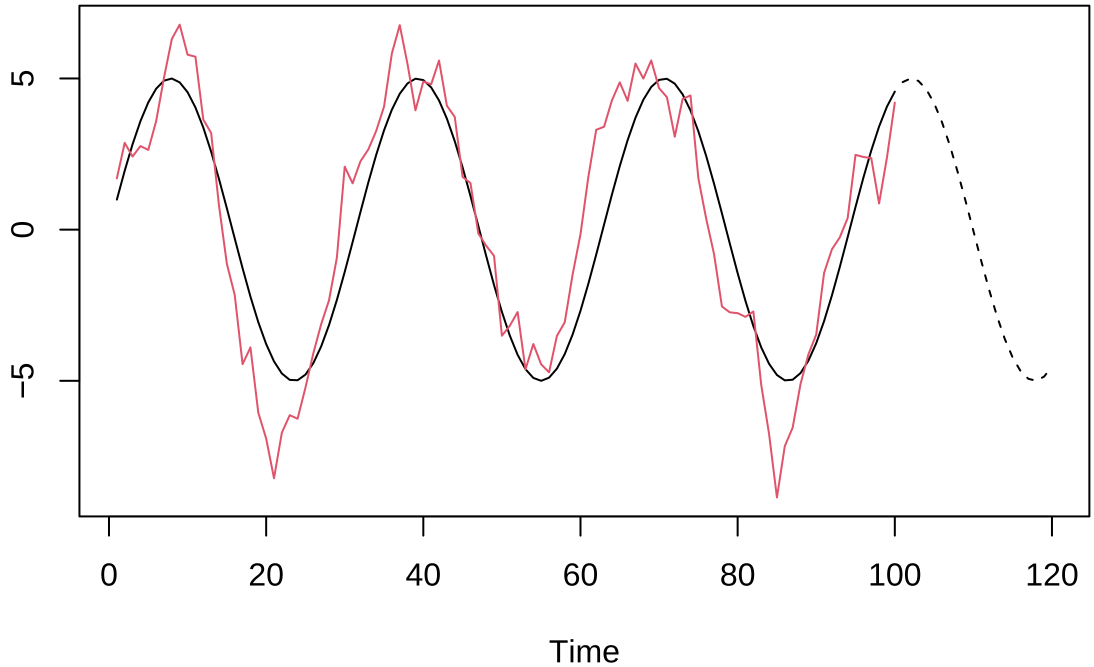


- Residuals indicate that there could be some weekly variation not accounted for.

1.8 Prediction

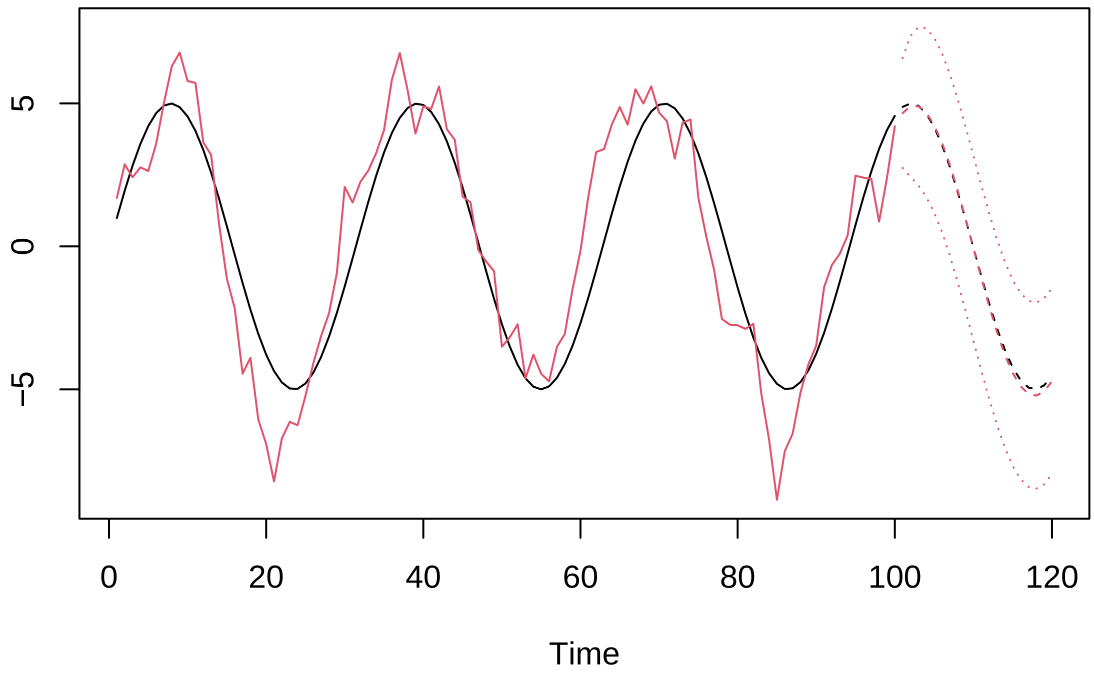
- Prediction can only be performed if we know the behavior of X_t for future time points, for example if we are able to control it.
- For the previous example we assume that the sine curve continues:

```
nnew = 20
xnew = lag(as.ts(5*sin(((n+1):(n+nnew))/5)),-n)
ts.plot(x,y,xnew,col=c(1,2,1),lty=c(1,1,2))
```



- We use the predict function.

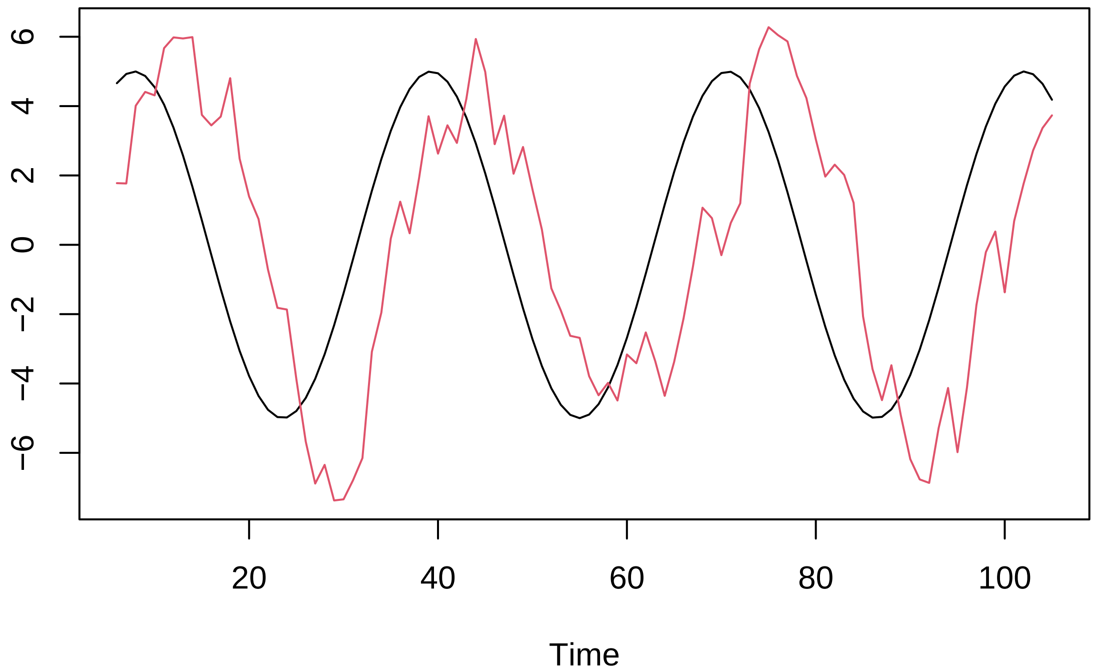
```
p = predict(mod,n.ahead=nnew,newxreg=xnew)
ts.plot(x,y,xnew,p$pred,p$pred+2*p$se,p$pred-2*p$se,col=c(1,2,1,2,2,2),lty=c(1,1,2,2,3,3))
```



1.9 An example with delay

- If we model the influence of X_t on Y_t, it may take some time before Y_t responds to a change in X_t.
  - Say the delay is k time steps.
- We want to model the effect of X_{t-k} on Y_t.
  - We may not know the delay k, so we may need to estimate it first.
- We simulate a dataset with a built-in delay, and then we model this afterwards.

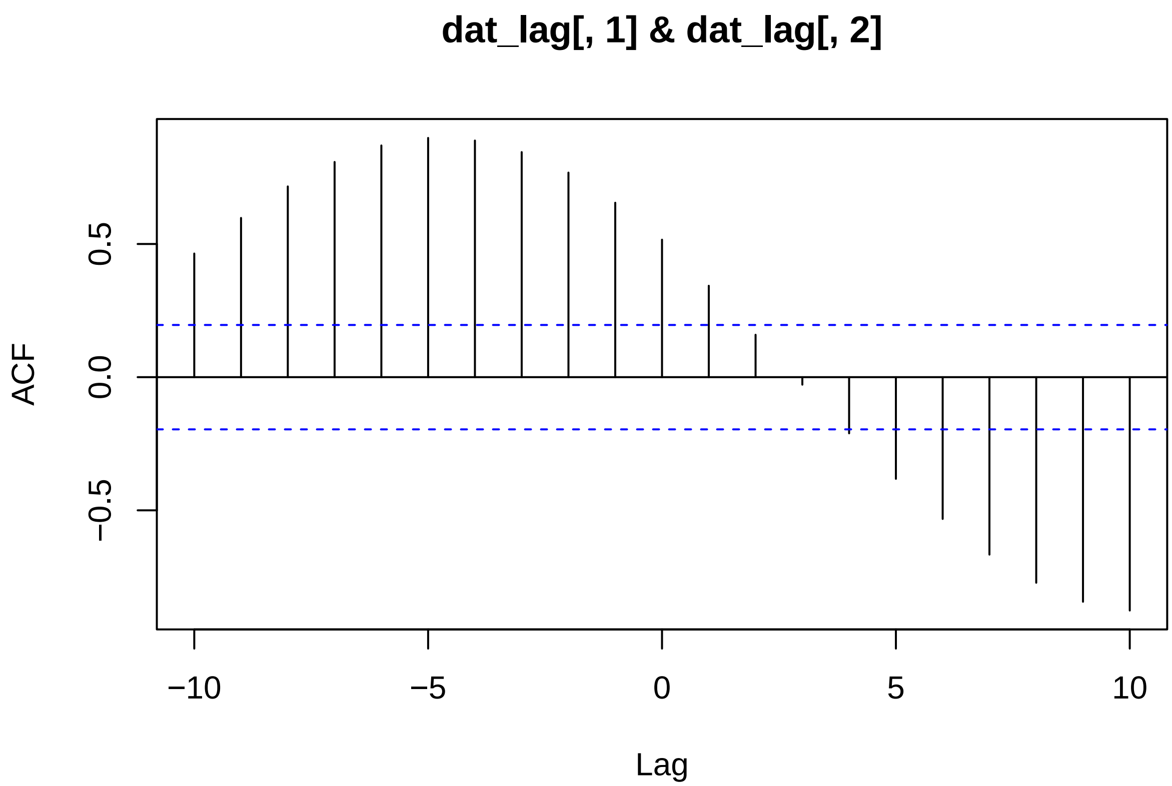
```
alpha = 0.5; gamma = 1; n = 100; delay = 5
x = as.ts(5*sin(1:(n+delay)/5))
eps = arima.sim(model=list(ar=alpha),n=n+delay)
y = gamma*lag(x,-delay)+eps
dat_lag = ts.intersect(x,y)
ts.plot(dat_lag[,1],dat_lag[,2],col=1:2)
```



1.10 The cross-correlation function

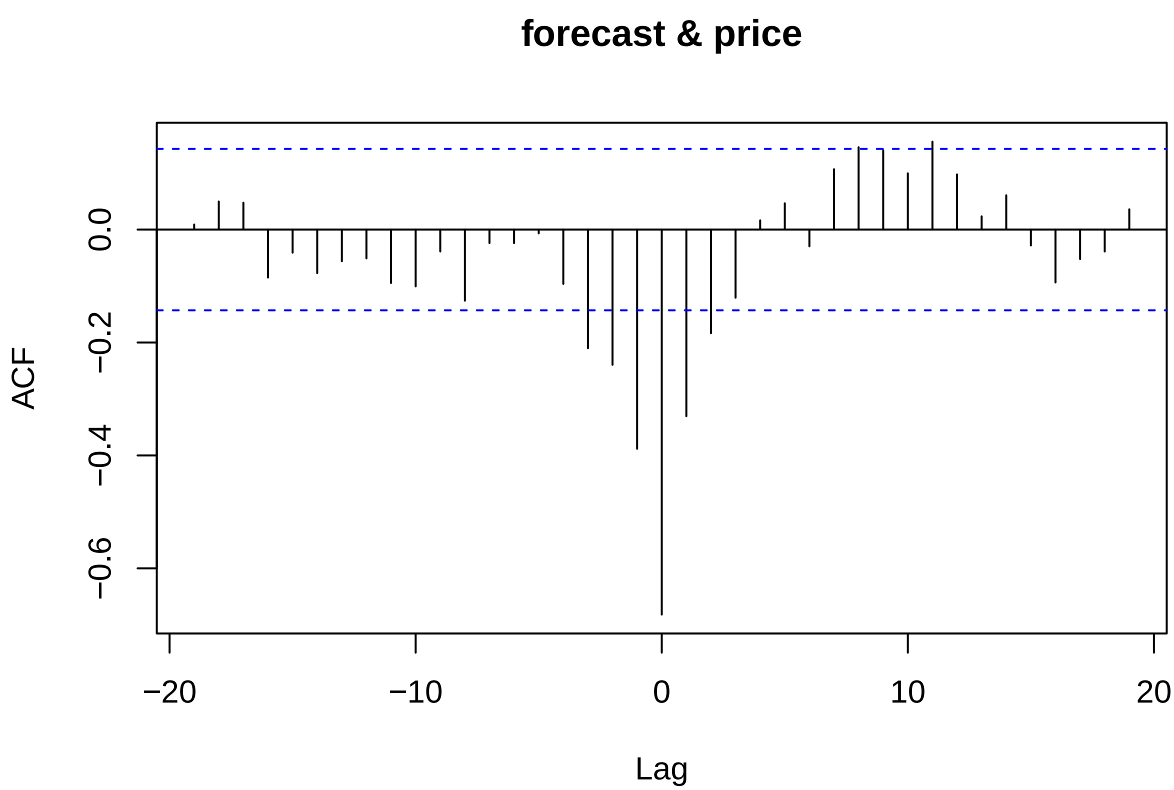
- The **cross correlation function** is used for checking the relation between two time series at different time points:
$$\rho_{xy}(t+k,t) = \mathrm{Cor}(X_{t+k}, Y_t).$$
- Values that are close to 1 or -1 indicate that the two time series are closely related if X_t is delayed by k time steps.
- Cross-correlation function for the simulated data

```
cc = ccf(dat_lag[,1],dat_lag[,2],lag.max=10)
```



- Cross-correlation function for the elspot data:

```
ccf(forecast,price)
```



1.11 Fitting models with lag

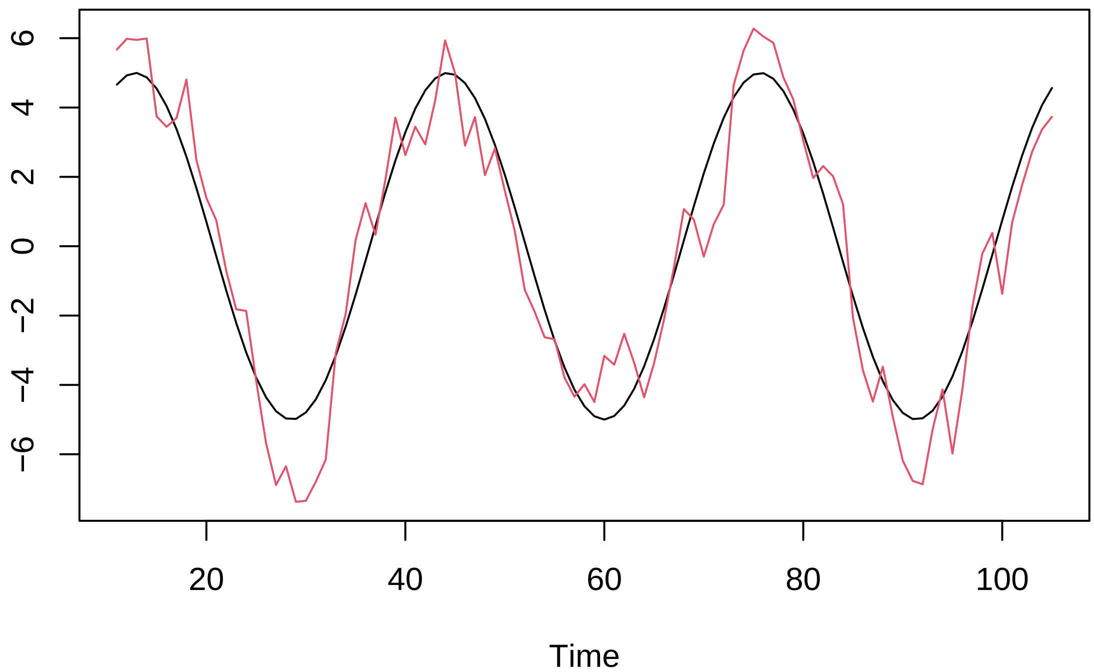
- We estimate the lag to be the one where the cross-correlation function is maximal:

```
estlag = cc$lag[which(cc$acf==max(abs(cc$acf)))]
estlag
```

```
## [1] -5
```

- Plotting the data with this lags can be useful to check the choice:

```
dat_shifted = ts.intersect(lag(as.ts(dat_lag[,1]),estlag),dat_lag[,2] )
ts.plot(dat_shifted[,1],dat_shifted[,2],col=1:2)
```



- We can now fit a model with this lag:

```
mod=arima(dat_shifted[,2],order=c(1,0,0),xreg=dat_shifted[,1]); mod
```

```
##
## Call:
## arima(x = dat_shifted[, 2], order = c(1, 0, 0), xreg = dat_shifted[, 1])
##
## Coefficients:
##          ar1  intercept  dat_shifted[, 1]
##       0.5938    -0.2047            1.0526
## s.e.  0.0820     0.2347            0.0615
##
## sigma^2 estimated as 0.8884:  log likelihood = -129.39,  aic = 266.79
```

1.12 ARMAX models

- An alternative way of including exogenous variables into an ARMA model is an ARMAX model.
- The $ARMAX(p, q, b)$ model is an $ARMA(p, q)$ model including b terms of an exogenous variable, i.e. it

is defined by

$$Y_t = \sum_{i=1}^{p} \alpha_i Y_{t-i} + \sum_{i=1}^{b} \gamma_i X_{t-i} + W_t + \sum_{i=1}^{q} \beta_i W_{t-i}$$

- Using the backshift operator, this can be written as

$$\alpha(B)Y_t = \gamma(B)X_t + \beta(B)W_t$$

  with $\alpha(B) = 1 - \sum_{i=1}^{p} \alpha_i B^i$, $\beta(B) = 1 + \sum_{i=1}^{q} \beta_i B^i$, and $\gamma(B) = \sum_{i=1}^{b} \gamma_i B^i$.
- Compare with the regression with ARMA noise:

$$\alpha(B)(Y_t - \gamma_0 - \gamma_1 X_t) = \beta(B)W_t \quad \Rightarrow \quad \alpha(B)Y_t = \alpha(B)(\gamma_0 + \gamma_1 \gamma X_t) + \beta(B)W_t$$

- The difference is only how the model includes the exogenous variable.
- It is mostly a matter of taste which kind of model you should choose.
- Only the regression with ARMA noise is included into R as standard.

2 Continuous time processes

2.1 Discrete vs. continuous time

There are two fundamentally different model classes for time series data.

- Discrete time stochastic processes
  - Variables given at equally spaced time points
- Continuous time stochastic processes
  - Variables that evolve over a continuous time scale

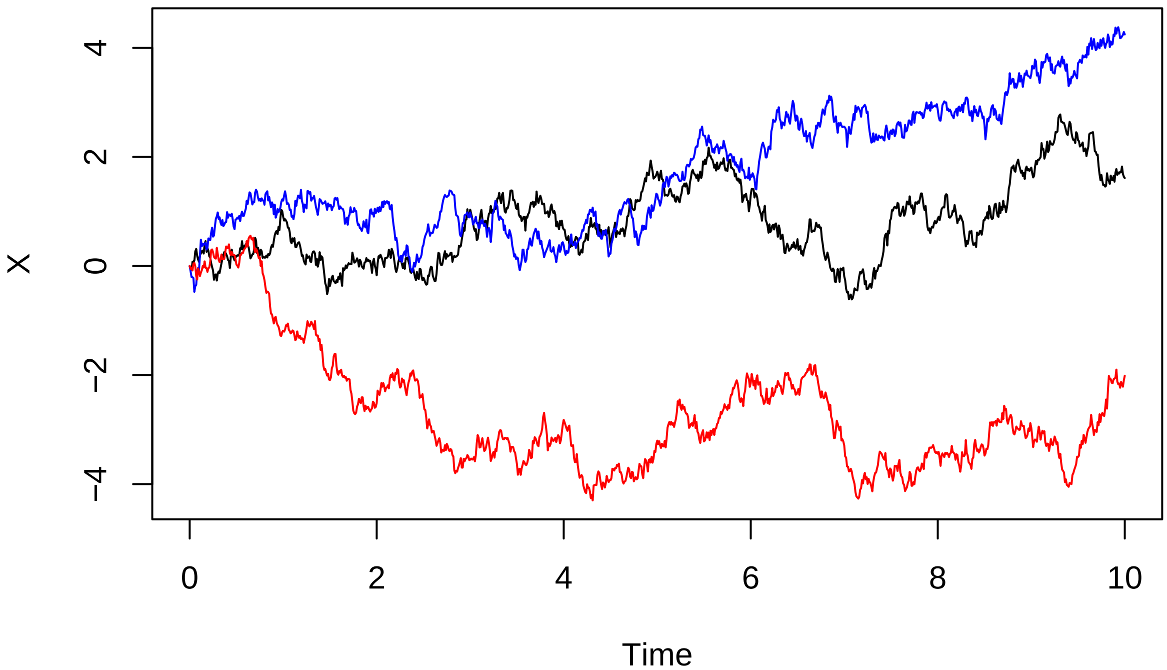
So far we have only looked at the discrete time case. We will finish todays lecture by looking a bit at the continuous time case, just to give you an idea of this topic.

2.2 Continuous time stochastic processes

- In this setup we see the underlying X_t as a continuous function of t for t in some interval $[0, T]$.
- In principle we imagine that there are infinitely many data points, simply because there are infinitely many time points between 0 and T.
- In practice we will always only have finitely many data points.
- But we can imagine that the real data actually contains all the data points. We are just not able to measure them (and to store them in a computer).
- With a model for all datapoints, we are - through simulation - able to describe the behaviour of data. Also between the observations.

2.3 The Wiener process

- A key example of a process in continuous time will be the so–called **Wiener process** or **Brownian motion**.
- Here are three simulated realizations (black, blue and red) of this process: here



- A Wiener process has the following properties:
  - It starts in 0: $B_0 = 0$.
  - It has independent increments: For $0 < s < t$ it holds that $B_t - B_s$ is independent of everything that has happened up to time s, that is B_u for all $u \leq s$.
  - It has normally distributed increments: For $0 < s < t$ it holds that the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$:

$$B_t - B_s \sim \texttt{norm}(0, t - s).$$

- The intuition of this process is that it somehow changes direction all the time: How the process changes after time s will be independent of what has happened before time s. So whether the process should increase or decrease after s will not be affected by how much it was increasing or decreasing before. This gives the very bumpy behaviour over time.

2.4 Stochastic differential equations

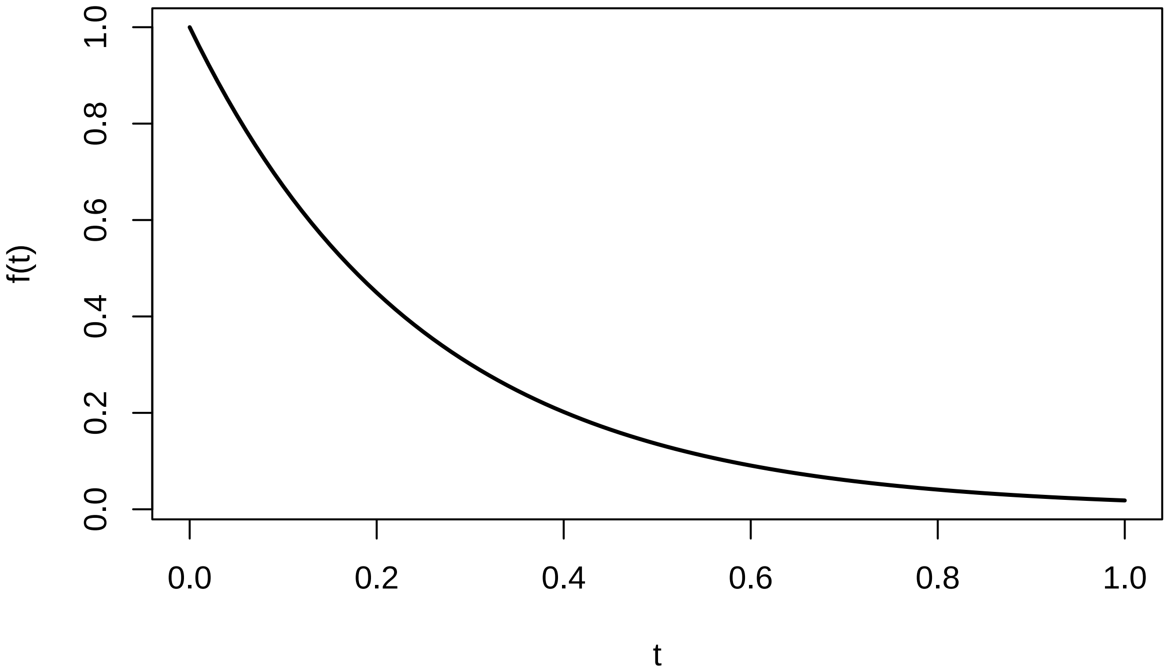
- A common way to define a continuous time stochastic process model is through a stochastic differential equation (SDE) which we will turn to shortly, but before doing so we will recall some basic things about ordinary differential equations.

- **Example:** Suppose f is an unknown differentiable function satisfying the differential equation

$$\frac{df(t)}{dt} = -4f(t)$$

with initial condition $f(0) = 1$. This equation has the solution

$$f(t) = \exp(-4t)$$



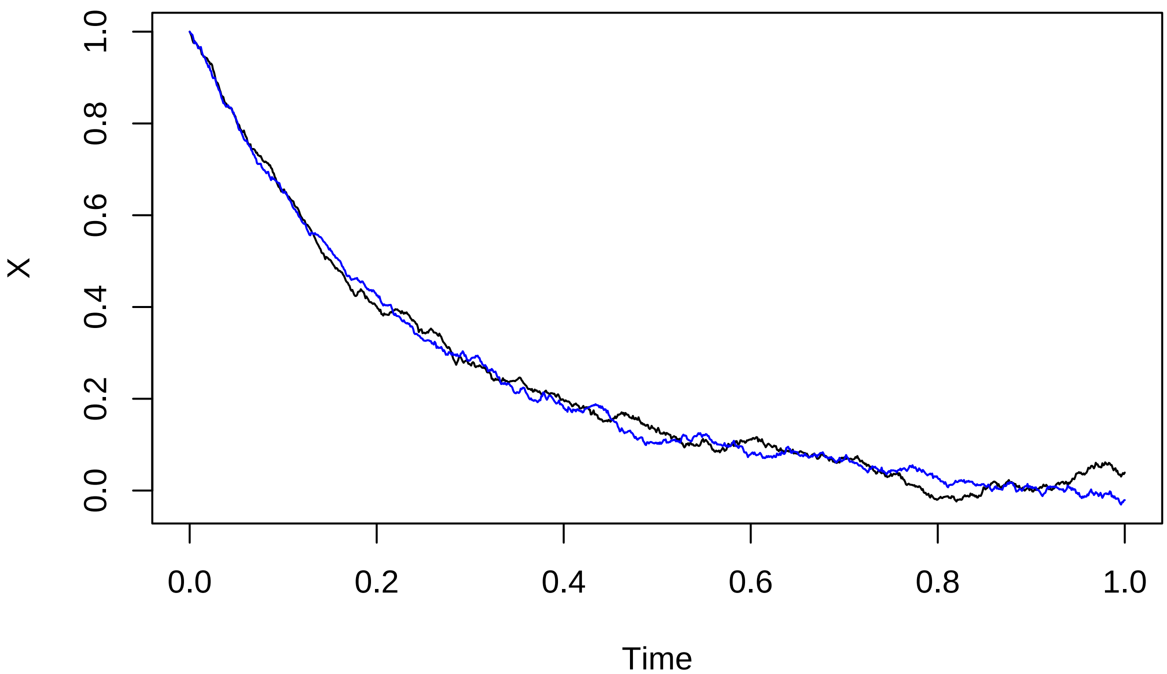
- With a slightly unusual notation we can rewrite this as

$$df(t) = -4 \cdot f(t)dt$$

- This equation has the following (hopefully intuitive) interpretation:
  - When time increases by a small amount dt (from t to $t+dt$) the value of f changes (approximately) by $-4f(t) \cdot dt$.
- So when t is increased, then f is decreased, and the decrease is proportional to the value of $f(t)$. That is why f decreases slower and slower, when t is increased.
- We say that the function has a **drift** towards zero, and this drift is determined by the value of the function.

2.5 Stochastic differential equations

- It will probably never be true that data behaves exactly like the exponentially decreasing curve on the previous slide.
- Instead we will consider a model, where some random noise from a Wiener process has been added to the growth rate. Two different (black/blue) simulated realizations can be seen below



- The type of process that is simulated above is described formally by the equation

$$dX_t = -4X_t dt + 0.1dB_t$$

- This is called a **Stochastic Differential Equation** (SDE), and the processes simulated above are called solutions of the stochastic differential equation.
- The SDE $dX_t = -4X_t dt + 0.1dB_t$ has two terms:
  - $-4X_t dt$ is the **drift term**.
  - $0.1dB_t$ is the **diffusion term**.
- The intuition behind this notation is very similar to the intuition in the equation $df(t) = -4 \cdot f(t)\, dt$ for an ordinary differential equation. When the time is increased by the small amount dt, then the process X_t is increased by $-4X_t\, dt$ AND by how much the process $0.1B_t$ has increased on the time interval $[t, t + dt]$.
- So this process has a **drift** towards zero, but it is also pushed in a random direction (either up or down) by the Wiener process (more precisely, the process $0.1B_t$)