ASTA

The ASTA team

Contents

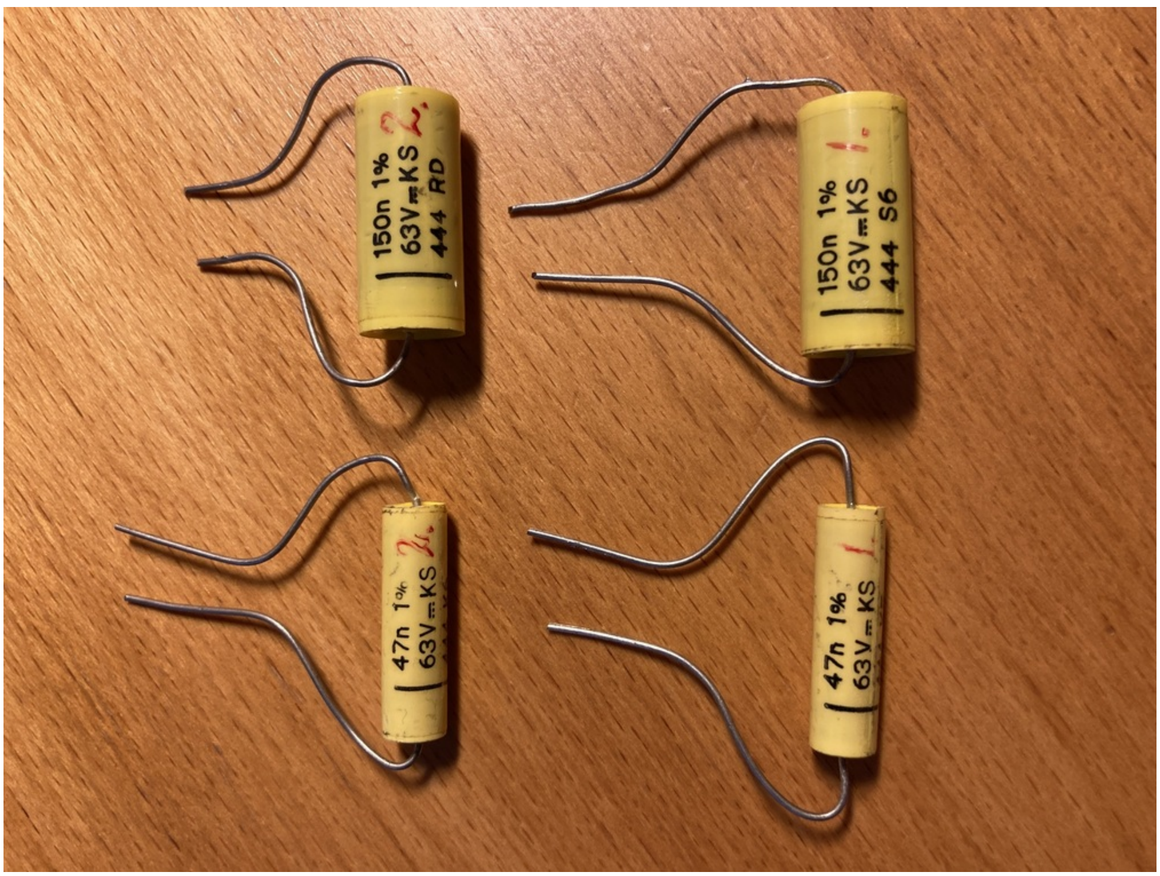
0.1 Sources of variation

Capacitors come with a nominal value for the capacitance.

- When capacitance is measured, we do not get exactly the nominal value.

We shall study 2 sources of variation:

- measurement variation due to random errors on a measuring device
- component variation due to random errors in the production process



0.2 Data from Peter Koch

Peter has done 100 independent measurements of the capacitance of each 4 of the displayed capacitors and one additional.

- Nominal values are 47, 47, 100, 150, 150 nF.
- All have a stated tolerance of 1%.

```r
load(url("https://asta.math.aau.dk/datasets?file=cap_1pct.RData"))
head(capDat, 4)
```

```
##   capacity nomval   sample
## 1    45.69     47 s_1_nF47
## 2    45.71     47 s_1_nF47
## 3    45.69     47 s_1_nF47
## 4    45.71     47 s_1_nF47
```

Here we see the first 4 measurements of the first capacitor with nominal value 47nF.

- Remark: The measured values are consistently below the nominal value minus the 1% tolerance: $47 - 0.47 = 46.53$.

```r
table(capDat$sample)
```

```
##
##  s_1_nF47  s_2_nF47 s_3_nF100 s_4_nF150 s_5_nF150
##       100       100       100       100       100
```

0.3 Relative errors

- Instead of considering the raw errors

$$\text{measuredValue - nominalValue},$$

  we will consider the relative error

$$\frac{\text{measuredValue - nominalValue}}{\text{nominalValue}}.$$

- A tolerance of 0.01 means that the relative error should be within ± 0.01.

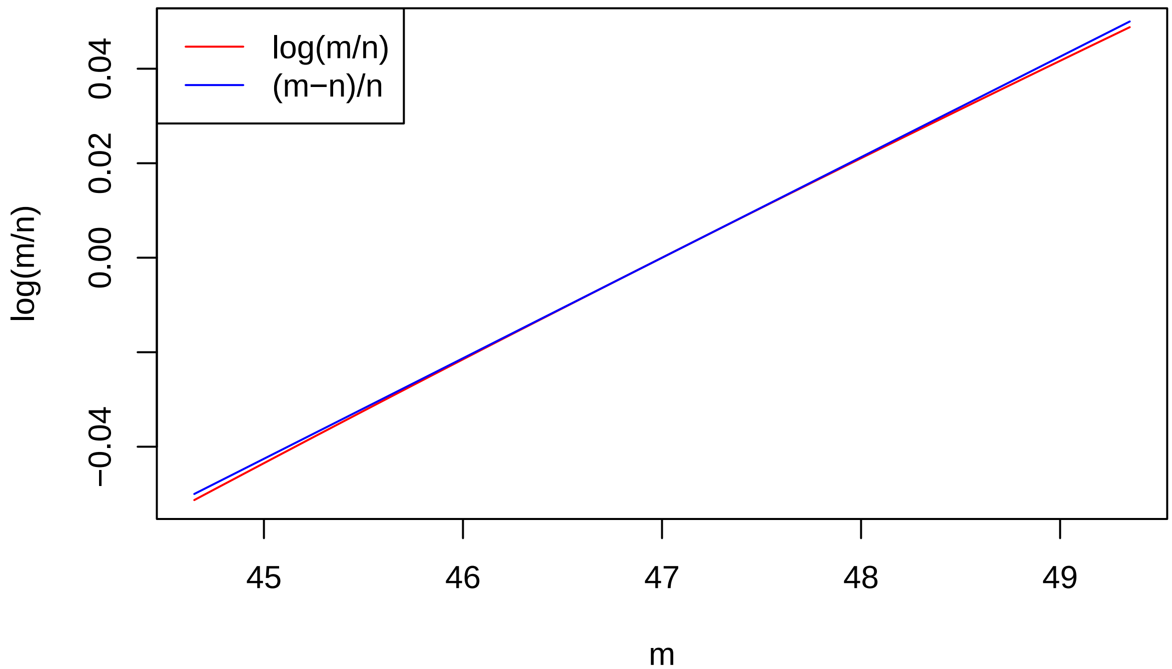
0.4 Approximation of the relative error

- Instead of looking at the relative error, we may look at the following approximation:

$$\text{lnError} = \ln\left(\frac{\text{measuredValue}}{\text{nominalValue}}\right) \approx \frac{\text{measuredValue} - \text{nominalValue}}{\text{nominalValue}}$$

- This is illustrated below with a nominal value of $n = 47$ and measured values of 47 plus/minus 5%.

```r
n <- 47
m <- seq(47-5*0.01*47, 47+5*0.01*47, length.out = 100)
plot(m, log(m/n), col = "red", type = "l")
lines(m, (m - n)/n, col = "blue", type = "l")
legend("topleft", legend = c("log(m/n)", "(m-n)/n"), lty = 1, col = c("red", "blue"))
```



0.5 Transformation of errors

- The approximation can be justified theoretically.
- Recall the linear approximation of a function:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

- If we take

$$x_0 = 1 \quad (1)$$
$$f(x) = \ln x \quad (2)$$
$$f'(x) = 1/x, \quad (3)$$

  we get

$$\ln(x) \approx \ln(x_0) + \frac{1}{x_0} \cdot (x - x_0) = x - 1.$$

- Suppose $x = m/n$. Then

$$\ln\left(\frac{m}{n}\right) \approx \frac{m}{n} - 1 = \frac{m-n}{n}$$

0.6 Transformed data

- We construct an extra `lnError` variable in the `capDat` dataset.

```
capDat = within(capDat, lnError <- log(capacity/nomval))
head(capDat, 2)
```

```
##   capacity nomval   sample     lnError
## 1    45.69     47 s_1_nF47 -0.02826815
## 2    45.71     47 s_1_nF47 -0.02783051
```

```r
tail(capDat, 2)
```

```
##     capacity nomval    sample     lnError
## 499    145.7    150 s_5_nF150 -0.02908558
## 500    145.6    150 s_5_nF150 -0.02977216
```

- The resolution on Peters capacitance meter is with 1-2 decimal(s) in the 47/150 nF range, which means that only a limited number of different values(3-18) are observed for each capacitor. This means that box-plots and histograms are non-informative.

0.7 Model considerations

- Let us have a look at a summary of the data:

```r
favstats(lnError~sample, data=capDat)
```

```
##      sample         min          Q1      median          Q3         max
## 1  s_1_nF47 -0.02958221 -0.02832287 -0.02804930 -0.02804930 -0.02783051
## 2  s_2_nF47 -0.02914399 -0.02783051 -0.02761176 -0.02761176 -0.02717441
## 3 s_3_nF100 -0.03521276 -0.03399638 -0.03386707 -0.03366020 -0.03334998
## 4 s_4_nF150 -0.02565975 -0.02446352 -0.02429269 -0.02429269 -0.02360987
## 5 s_5_nF150 -0.03045921 -0.02977216 -0.02908558 -0.02908558 -0.02908558
##          mean           sd   n missing
## 1 -0.02832518 0.0005062160 100       0
## 2 -0.02786346 0.0005171088 100       0
## 3 -0.03398306 0.0005057586 100       0
## 4 -0.02453879 0.0005870180 100       0
## 5 -0.02947702 0.0005543930 100       0
```

- All measurements are more than 2.3% below the nominal value.
- This must be due to a systematic error on the meter.

0.8 Sources of variation

- We now have three sources of error:
  - Systematic errors of the measurement device
  - Production errors in the individual capacitors
  - Random measurement errors
- This leads us to consider the model

$$\ln\left(\frac{\text{measuredValue}}{\text{nominalValue}}\right) = \text{systematicError} + \text{productionError} + \text{measurementError}.$$

0.9 Statistical model

- We have the model:

$$\ln\left(\frac{\text{measuredValue}}{\text{nominalValue}}\right) = \text{systematicError} + \text{productionError} + \text{measurementError}$$

- We may write the model mathematically as

$$Y_{ij} = \mu + A_i + \varepsilon_{ij}$$

  where

  - Y_{ij} is the log error measurement (jth measurement from the ith capacitor)
  - $i = 1, \ldots, k$ is the number of the capacitor

- $j = 1, \ldots, n$ is the number of the observation for that capacitor
  - $k = 5$ is the total number of capacitors
  - $n = 100$ is the number of repetitions for each capacitor
  - μ is the systematic error on the meter
  - A_i is the random production error
  - ε_{ij} is the random measurement error
- We make the following assumptions:
  - The production error A_i is normally distributed with mean 0 and variance σ_α^2,
  - The measurement error ε_{ij} is normally distributed with mean 0 and variance σ^2.
- This is called a **random effects model**, see [WMMY] Chapter 13.11.

0.10 Estimation of systematic error

- The systematic error is simply estimated by the sample mean

$$\hat{\mu} = \bar{y}_{..}$$

  - The two dots indicate that we take the average over all observations from all capacitors.

```
muhat <- mean(capDat$lnError)
muhat
```

```
## [1] -0.0288375
```

- The meter systematically reports a value, which is estimated to be 2.88% too low.

0.11 Estimation of random error

- We now try to estimate the variance σ_α^2 of the production error and the variance of the random measurement error σ^2.
- We need two types of sum of squares:
- SSA (*sum of squares between groups*) measures how much the sample means for the individual capacitors $\bar{y}_{i.}$ deviate from the total sample mean $\bar{y}_{..}$

$$SSA = n \sum_i (\bar{y}_{i.} - \bar{y}_{..})^2$$

- SSE (*sum of squares within groups*) measures how much the individual measurements deviate from the sample mean of the capacitor they were measured on:

$$SSE = \sum_{ij} (y_{ij} - \bar{y}_{i.})^2$$

- Intuitively, SSA is closely related to the variance of the production error σ_α^2, while SSE is closely related to the variance of the random measurement error σ^2.

0.12 Fit

- The sum of squares may be found from:

```
fit <- lm(lnError ~ sample, data = capDat)
anova(fit)
```

```
## Analysis of Variance Table
##
## Response: lnError
##            Df    Sum Sq    Mean Sq F value    Pr(>F)
## sample      4 0.0046576 0.00116440  4067.4 < 2.2e-16 ***
## Residuals 495 0.0001417 0.00000029
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- We can extract the sum of squares as follows

```
SS <- anova(fit)$`Sum Sq`
SSA <- SS[1]
SSE <- SS[2]
SSA
```

```
## [1] 0.004657588
```

```
SSE
```

```
## [1] 0.0001417076
```

0.13 Solution

- One may show (see [WMMY] Theorem 13.4):

$$E(SSA) = (k-1)\sigma^2 + n(k-1)\sigma_\alpha^2$$
$$E(SSE) = k(n-1)\sigma^2$$

- Using the approximations

$$E(SSA) \approx SSA, \qquad E(SSE) \approx SSE$$

  we obtain the estimates

$$\hat{\sigma}^2 = \frac{1}{(n-1)k} SSE = \frac{1}{99 \cdot 5} \cdot 0.0001417 = 2.86 \cdot 10^{-7}$$

$$\hat{\sigma}_\alpha^2 = \frac{1}{n(k-1)} SSA - \frac{\hat{\sigma}^2}{n} = \frac{1}{100 \cdot (5-1)} \cdot 0.0046576 - \frac{2.86 \cdot 10^{-7}}{100} = 1.16 \cdot 10^{-5}$$

0.14 Summing up

- The meter has an estimated systematic error of $\hat{\mu} = -2.88\%$.
- The estimated standard deviation of the meter is $\hat{\sigma} = \sqrt{2.86 \cdot 10^{-7}} = 0.0534\%$.
- The estimated standard deviation of the production error is $\hat{\sigma}_\alpha = \sqrt{1.16 \cdot 10^{-5}} = 0.341\%$.
- Since 99.7% (practically all) of all observations fall within $\pm 3 \cdot \sigma_\alpha$ from 0, we have that the production error falls within

$$\pm 3 \cdot 0.341\% = 1.02\%$$

  of the nominal value, which is in accordance with the tolerance of 1%.

- The total estimated variance of the log error is

$$\hat{\sigma}_\alpha^2 + \hat{\sigma}^2 = 1.16 \cdot 10^{-5} + 2.86 \cdot 10^{-7} = 1.19 \cdot 10^{-5}.$$

  - The variance is clearly dominated by the production error.

- Note that especially the estimate $\hat{\sigma}_\alpha$ is quite uncertain, since we only have measurements from 5 capacitors.

0.15 Test of no random effect

- We have the possibility of testing the hypothesis

$$H_0 : \sigma_\alpha = 0.$$

- The formulas for $E(SSA)$ and $E(SSE)$ were

$$E(SSA) = (k-1)\sigma^2 + n(k-1)\sigma_\alpha^2$$

$$E(SSE) = k(n-1)\sigma^2.$$

- Under H_0, this is means that

$$\frac{1}{k-1}E(SSA) = \frac{1}{k(n-1)}E(SSE) = \sigma^2.$$

- Under H_0, the F statistic

$$F_{obs} = \frac{\frac{SSA}{k-1}}{\frac{SSE}{k(n-1)}}$$

  has an F-distribution with degrees of freedom $df_1 = k-1$ and $df_2 = k(n-1)$.
  - Large values are critical for the null-hypothesis.
- In the capacitor dataset $F_{obs} = 4067.4$, which is highly significant (p-value close to 0).
  - Our capacitors do have some production errors.

0.16 Coefficient of variation

- Let X be a random variable with mean μ and standard deviation σ.
- If we are interested in relative variation, it is common to look at the **coefficient of variation**

$$CV(X) = \frac{\sigma}{\mu}$$

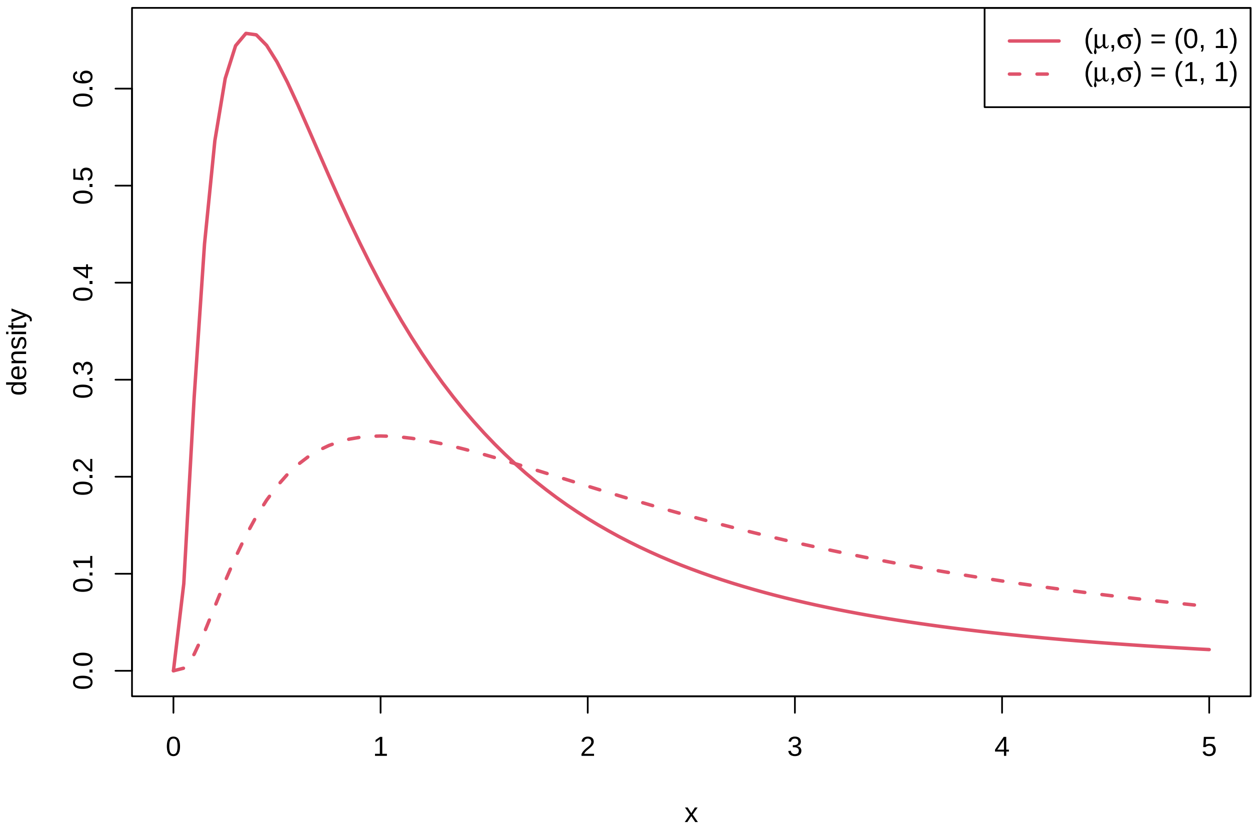
  - Standard deviation relative to the mean
  - Unit-free
- If X is normal, then 95% of our measurements are within

$$\mu \pm 2 \cdot \sigma = \mu \pm 2 \cdot \mu \cdot CV(X) = \mu(1 \pm 2 \cdot CV(X)).$$

- If e.g. $CV(X) = 0.05$, it means that 95% of all observations are within $2 \cdot 0.05 = 10\%$ of the mean.

0.17 The lognormal distribution

- In the preceeding analysis, we assumed that the log-transformed errors had a normal distribution.
- Let X be a random variable and $Y = \ln(X)$.
- We say that X has a **lognormal distribution** if Y has a normal distribution with - say - mean μ and standard deviation σ.
- Here are some plots of the density of the lognormal distribution:



0.18 Coefficient of variation for lognormal distribution

- Suppose X has a log-normal distribution, so that $Y = \ln(X)$ has a normal distribution with mean μ and standard deviation σ.
- Then the mean and variance are given by (Theorem 6.7 of [WMMY]):

$$E(X) = \exp(\mu + \sigma^2/2)$$

$$Var(X) = \exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)$$

- The coefficient of variation is then

$$CV(X) = \frac{\sqrt{Var(X)}}{E(X)} = \frac{\sqrt{\exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)}}{\exp(\mu + \sigma^2/2)} = \sqrt{\exp(\sigma^2) - 1}$$

- In Peter's data we estimated the variance of the ln error to $\hat{\sigma}_\alpha^2 = 1.16 \cdot 10^{-5}$, which means that the estimated CV of the capacity measurement is

$$\widehat{CV}(X) = \sqrt{\exp\left(1.16 \cdot 10^{-5}\right) - 1} = 0.341\%.$$

0.19 Linear calibration

- In our previous analysis, we assumed, that the systematic error on the meter did not depend on nominal value.

$$\ln\left(\frac{\text{measuredValue}}{\text{nominalValue}}\right) = \text{meterError} + \text{randomError}$$

- To check this assumption consider the linear model

$$\ln(\text{measuredValue}) = \alpha + \beta \cdot \ln(\text{nominalValue}) + \varepsilon.$$

- Note that the previously considered model corresponds to $\beta = 1$.

0.20 Linear calibration fit

- We fit the linear model:

```
fit <- lm(log(capacity) ~ log(nomval), data = capDat)
summary(fit)
```

```
##
## Call:
## lm(formula = log(capacity) ~ log(nomval), data = capDat)
##
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0064121 -0.0010784  0.0007315  0.0013879  0.0050839
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.0300145  0.0011907  -25.21   <2e-16 ***
## log(nomval)  1.0002636  0.0002648 3776.74   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.003101 on 498 degrees of freedom
## Multiple R-squared:      1,  Adjusted R-squared:      1
## F-statistic: 1.426e+07 on 1 and 498 DF,  p-value: < 2.2e-16
```

- The slope looks close to 1.
- We may test the null-hypothesis $H_0 : \beta = 1$.

$$t_{obs} = \frac{1.0002636 - 1}{0.0002648} = 0.995.$$

  This yields a p-value of around 32%.

  - It is a bit dubious to model a linear relationship with only 3 nominal values.
  - Also note that we have correlated measurements, since several measurements are made on the same capacitors.

0.21 Calibrated values

- If we stick to the linear calibration model, it is sensible to correct our measured errors according to the calibration of the meter.
- We have the model:

$$\text{measuredValue} = \alpha + \beta * \text{nominalValue}$$

- We compute the calibrated values

$$\text{calibratedValue} = (\text{measuredValue} - \alpha)/\beta$$

- We estimate the coefficients α and β and calibrate the measurements.

```
ab = coef(fit)
ab
```

```
## (Intercept) log(nomval)
## -0.03001454  1.00026359
```

```r
capDat$lnError_c = (capDat$lnError - ab[1])/ab[2]
head(capDat)
```

```
##   capacity nomval   sample     lnError   lnError_c
## 1    45.69     47 s_1_nF47 -0.02826815 0.001745930
## 2    45.71     47 s_1_nF47 -0.02783051 0.002183452
## 3    45.69     47 s_1_nF47 -0.02826815 0.001745930
## 4    45.71     47 s_1_nF47 -0.02783051 0.002183452
## 5    45.70     47 s_1_nF47 -0.02804930 0.001964715
## 6    45.69     47 s_1_nF47 -0.02826815 0.001745930
```

1 Lot variation

- Picture of a “lot” of capacitors.
- The word lot is used to identify several components produced in a single run.
  - A run is a production series limited to a given time interval and fixed production parameters.
- We expect components from the same lot to be more similar.
- Peter Koch has tested 269 of the capacitors in the displayed lot (one measurement for each).

```r
Cap220=read.csv(url("https://asta.math.aau.dk/datasets?file=capacitor_lot_220_nF.txt"))[,1]
summary(Cap220)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##   197.2   204.8   207.9   207.9   210.9   218.6
```

2 Testing for log normality

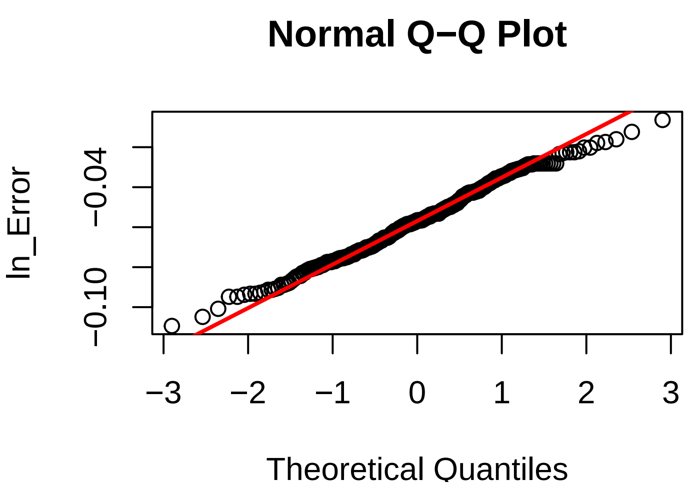
2.1 Log normality

- Last time we assumed log normality of the relative measurements:

$$\ln\left(\frac{\text{measuredValue}}{\text{nominalValue}}\right) \sim \text{norm}(\mu, \sigma).$$

- The data we considered last time did not allow us check this assumtion.
- We have seen that normality can be checked with a qqplot (lecture 1.3, [WMMY] Sec. 8.8).

```
Cap220=read.csv(url("https://asta.math.aau.dk/datasets?file=capacitor_lot_220_nF.txt"))[,1]
ln_Error=log(Cap220/220)
qqnorm(ln_Error,ylab="ln_Error")
qqline(ln_Error,lwd=2,col="red")
```



- The qq-plot supports normality of `ln_Error`.

2.2 Testing normality

- One can also make a test of the null-hypothesis

$$H_0: \text{ the population has a normal distribution.}$$

- There are several tests of normality.
- Two of these are considered in [WMMY] Section 10.11:
  - Gearys test
  - goodness of fit

2.3 Gearys test

- Consider a sample $X_1, \dots, X_n$ from a population.
- We may estimate of the standard deviation σ of the population:

$$S_0 = \sqrt{\frac{1}{n}\sum_i (X_i - \bar{X})^2}$$

  - S_0 is *always* a good estimator of the population standard deviation σ - no matter the form of the population distribution.
- Next consider

$$S_1 = \sqrt{\frac{\pi}{2}} \sum_i |X_i - \bar{X}|/n$$

  - This is a good estimator of σ, **if** the population is normal.
  - Otherwise, it will over- or underestimate σ depending on the form of the population distribution.

2.4 Gearys test

- If the population distribution is normal, we expect that

$$U = \frac{S_1}{S_0}$$

  is close to 1.

- Under the null-hypothesis,

$$Z = \frac{\sqrt{n}(U-1)}{0.2661}$$

  is approximately standard normally distributed when n is large.

- That is, with a significance level of 5%, we reject the null-hypothesis if $|z_{obs}| > 1.96$.

- We can do all the computations in R.

```
mln_E=mean(ln_Error)
s1=sqrt(mean((ln_Error-mln_E)^2))
s0=sqrt(pi/2)*mean(abs(ln_Error-mln_E))
u=s1/s0
z_obs=sqrt(length(ln_Error))*(u-1)/0.2661
z_obs
```

```
## [1] -1.383383
```

- We do not reject the null-hypothesis.
- Hence there is no evidence of non-normality.

2.5 Goodness of fit - die example

- Goodness of fit is a general method for investigating whether a sample comes from a specific distribution.

- Before considering test for normality, we consider a simpler example (see [WMMY] Sec. 10.11).

- Suppose we roll a die. We have the null-hypothesis that the die is fair, i.e. the probabilities of the outcomes $(1, 2, 3, 4, 5, 6)$ are

$$(1/6, 1/6, 1/6, 1/6, 1/6, 1/6).$$

- Rolling the die 120 times, we expect the frequencies

$$(20, 20, 20, 20, 20, 20)$$

- Actually we observe the frequencies

$$(20, 22, 17, 18, 19, 24)$$

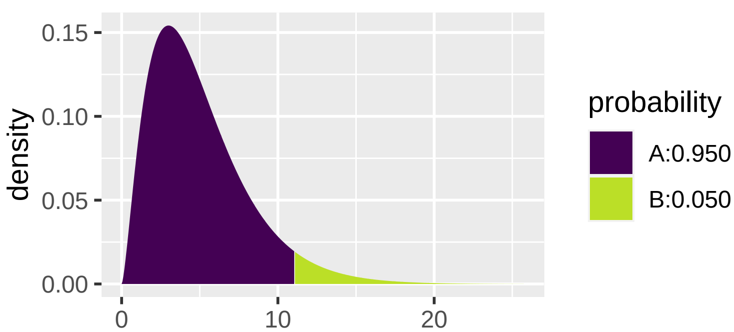
- The distance between observed and expected frequencies is measured by

$$X^2 = \sum \frac{(\text{ObservedFrequencies - ExpectedFrequencies})^2}{\text{ExpectedFrequencies}}$$

2.6 Goodness of fit - die example

- **If** the null-hypothesis is true (the die is fair), then
  - X^2 has a chi-square distribution (Lecture 1.4, [WMMY] Chapter 6.7) with df=k-1=5 degrees of freedom, where $k = 6$ is the number of possible outcomes.
  - large values of X^2 are critical for the null-hypothesis.
- For the example on the previous slide:
  - $x^2_{obs} = 1.7$

```
critical_value <- qdist("chisq", .95, df = 5)
```




```
critical_value
```

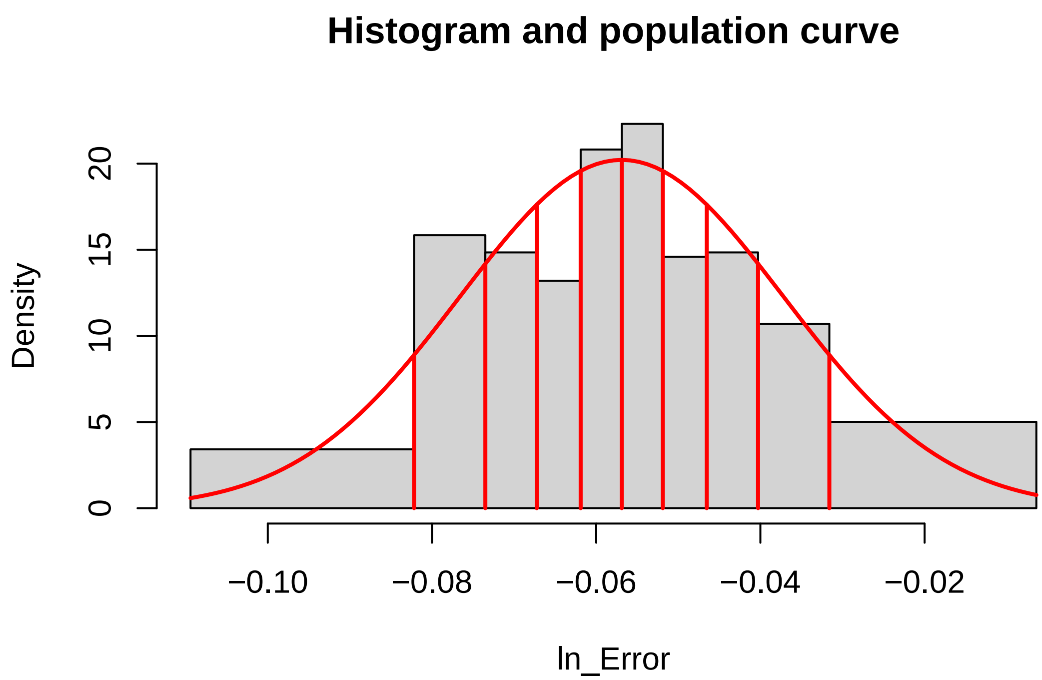
```
## [1] 11.0705
```

- At 5% significance level the critical value is 11.07, so there is no evidence against the null-hypothesis of a fair die.

2.7 Goodness of fit - normal distribution

- We assume that `ln_Error` is a sample from a normal distribution.
- We estimate its mean and standard deviation by the sample mean and sample standard deviation
- We divide the population distribution into 10 bins with equal probabilities p=10%.
  - The number of bins could be changed.
  - The bins should be so large, that the expected frequencies in each is at least 5.

```
m <- mean(ln_Error)
s <- sd(ln_Error)
breaks <- qnorm((0:10)/10, m, s)
```



- Area in each bin of the red population curve is 0.1
- As the sample size is 269 we obtain that the expected frequency is $269 * 0.1 = 26.9$ in each bin.
  - This is clearly above 5

2.8 Goodness of fit - normal distribution

- Observed frequecies:

```
observed <- table(cut(ln_Error, breaks))
names(observed) <- paste("bin", 1:10, sep = "")
observed
```

```
##  bin1  bin2  bin3  bin4  bin5  bin6  bin7  bin8  bin9 bin10
##    25    37    25    19    28    30    21    25    25    34
```

- We compute the X^2 statistic:

```
chisq_obs <- sum((observed-26.9)^2)/26.9
chisq_obs
```

```
## [1] 10.21933
```

- The degrees of freedom is the number of bins minus 3 (number of parameters + 1), i.e. df = 10-3 = 7.

2.9 Goodness of fit - normal distribution

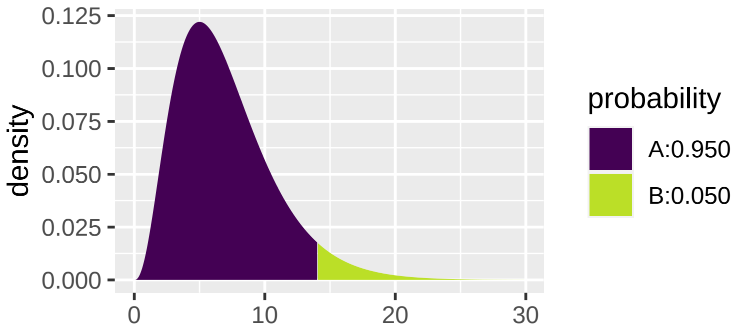
- We had computed the value of X^2

```
chisq_obs
```

```
## [1] 10.21933
```

- We find the critical value

```
critical_value <- qdist("chisq", .95, df = 7)
```




```
critical_value
```

```
## [1] 14.06714
```

- Since X^2 is smaller than the critical value, we do not reject the null-hypothesis
- We could also have used the p-value

```
p_value <- 1 - pchisq(chisq_obs, 7)
p_value
```

```
## [1] 0.1764812
```

- We do not reject normality at level 5%.

2.10 Other tests of normality

- There are many other tests of normality.
- We mention one of the most commonly used tests: Shapiro-Wilks.
- It is standard in `R`.
- We do not treat the details, but the test statistic is somewhat like a correlation for the qq-plot.
  - If the "correlation is far from 1", we reject normality.

```
shapiro.test(ln_Error)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  ln_Error
## W = 0.99255, p-value = 0.1971
```

- With a p-value of 19.71%, we do not reject normality, if we test on level 5%.

3 Sources of variation

- In lecture 4.1 we discussed 3 sources of variation:
  - systematic measurement error
  - random measurement variation
  - production variation
- Generally it is relevant to decompose the production variation in 2 components:
  - variation within lot, i.e. the variation around the lot mean

– variation between lots, i.e. the variation of the lot means.

3.1 The general model

- The completely general model would be:

$$\text{measuredValue} = \text{systematicError} + \text{lotError}$$
$$+\text{componentError} + \text{measurementError}$$

- In mathematical notation
$$Y_{k,i,j} = \mu + L_k + C_{k,i} + \varepsilon_{k,i,j}$$
  where
  - k is the number of the lot
  - i is the number of the component in lot k
  - j is the number of the measurement on component (k, i).
- The errors are assumed random and normal
  - Lot errors $L_k \sim norm(0, \sigma_l)$
  - Errors on individual component within lot $C_{k,i} \sim norm(0, \sigma_c)$
  - Measurement errors $\varepsilon_{k,i,j} \sim norm(0, \sigma_m)$

3.2 Model for our data

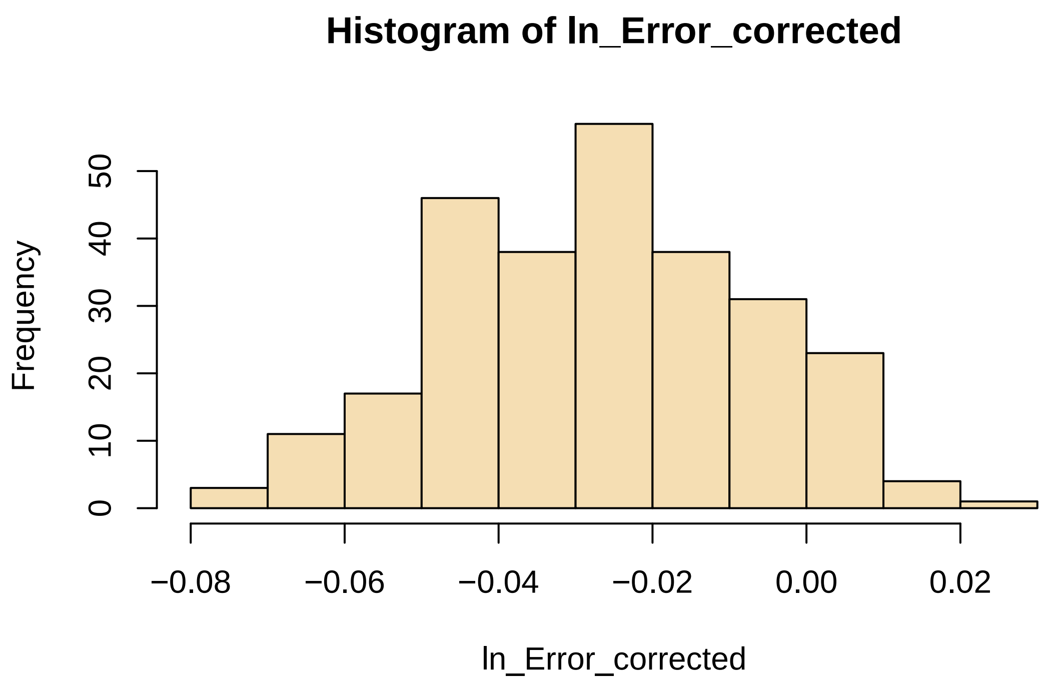
- As we have one lot only, we cannot identify the variation between lots.
  - We will consider the lot mean as fixed number μ_l
- We only have one measurement on each component
- The model for our data reduces to (since $k = 1$ and $j = 1$ we omit them from notation)
$$Y_i = \mu + \mu_l + C_i + \varepsilon_i$$
  where
  - $i = 1, \ldots, 269$ is observation number
  - μ is systematic measurement error
  - μ_l is systematic lot error
  - $C_i \sim norm(0, \sigma_c)$ is variation within lot
  - $\varepsilon_i \sim norm(0, \sigma_m)$ is measurement error

---

3.3 Linear calibration

- In lecture 4.1 we developed a linear calibration to eliminate the systematic measurement error.
- To remove the systematic measurement error, we apply this calibration to our new dataset.

```
load("ab.RData")
ln_Error_corrected <- (ln_Error-ab[1])/ab[2]
hist(ln_Error_corrected, breaks = "FD", col = "wheat")
```



3.4 Model for calibrated data

- After calibration, we will assume that the systematic measurement is zero, leaving us with the model for the calibrated values:

$$Y_i = \mu_l + C_i + \varepsilon_i$$

  where
  - $i = 1, \ldots, 269$ is observation number
  - μ_l is systematic lot error
  - $C_i \sim norm(0, \sigma_c)$ is variation within lot
  - $\varepsilon_i \sim norm(0, \sigma_m)$ is measurement error
- We are this left with a normally distributed sample with
  - mean μ_l
  - variance $\sigma_c^2 + \sigma_m^2$

3.5 Estimate of parameters

- Estimate of μ_c

```
myl <- mean(ln_Error_corrected)
myl
```

```
## [1] -0.02686793
```

- That is, the systematic lot error is around -2.7%.

- Estimate of $\sigma_m^2 + \sigma_c^2$

```
var(ln_Error_corrected)
```

```
## [1] 0.0003892828
```

- That is $s_m^2 + s_c^2 = 3.9 \cdot 10^{-4}$.

- In lecture 4.1 we estimated $s_m^2 = 0.29 \cdot 10^{-6}$ and hence $s_c^2 = 3.9 \cdot 10^{-4}$

$$s_c = \sqrt{3.9 \cdot 10^{-4}} = 0.02$$

- 3 sigma limits for the corrected lot values:

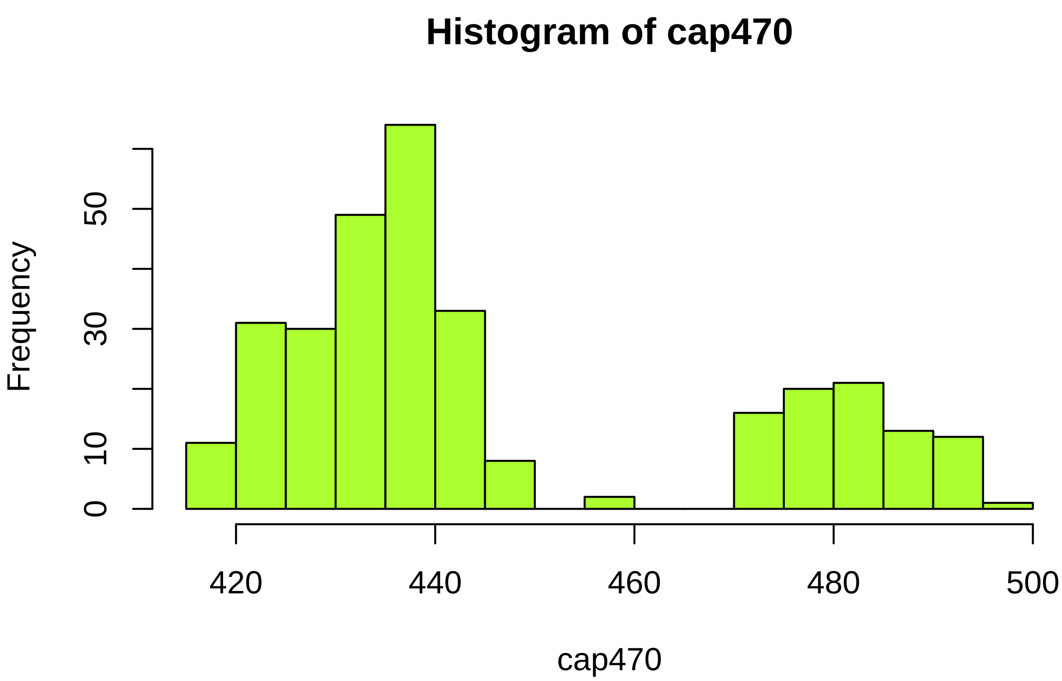
$$-2.7\% \pm 3 \cdot 2.0\% = [-8.7; 3.3]\%$$

clearly respecting the 10% tolerance.

4 Mixture of lots

- Peter has also tested 311 capacitors with nominal value 470 nF

```
cap470 <- read.table(url("https://asta.math.aau.dk/datasets?file=capacitor_lot_470_nF2.txt"))[, 1]
hist(cap470, breaks = 15, col = "greenyellow")
```

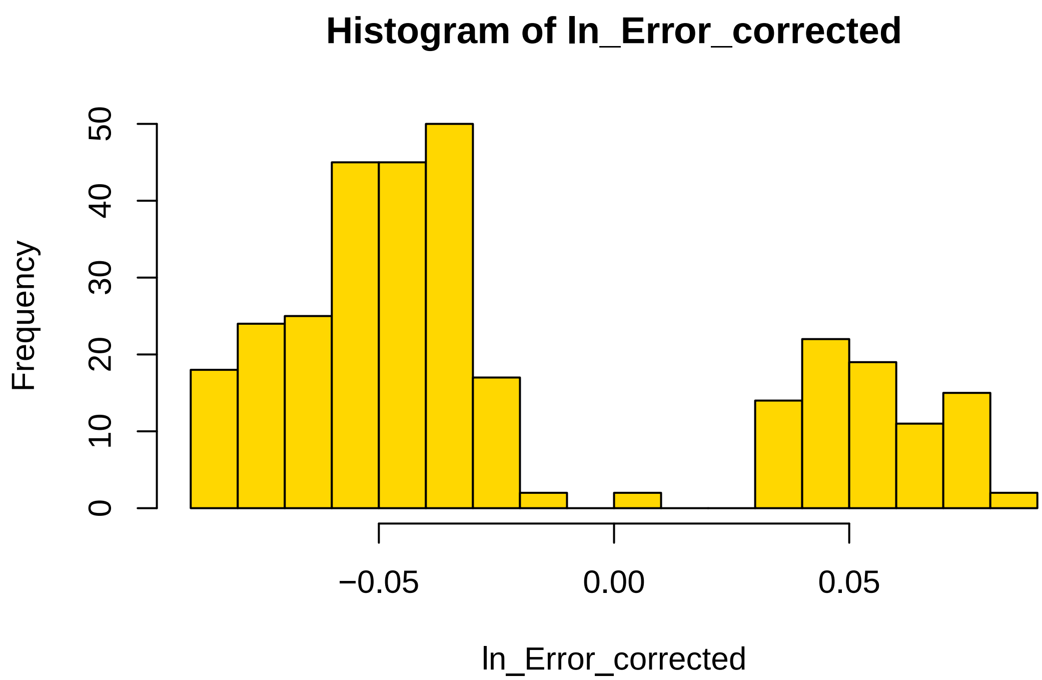


- Consulting Peter, it turned out, that his box of capacitors contained components from 2 different lots.

4.1 Transforming

- We ln-transform and calibrate:

```
ln_Error <- log(cap470/470)
ln_Error_corrected <- (ln_Error-ab[1])/ab[2]
hist(ln_Error_corrected, breaks = 15, col = "gold")
```




```
range(ln_Error_corrected)
```

```
## [1] -0.08888934  0.08323081
```

4.2 Mixture model

- We assume that the ln_Error
  - is normal with mean μ_1 if the component is from lot 1
  - is normal with mean μ_2 if the component is from lot 2
  - both distributions have variance $\sigma^2 = \sigma_m^2 + \sigma_l^2$
  - the probability of coming from lot 1 is p
- So we have 4 unknown parameters: $(\mu_1, \mu_2, \sigma, p)$.
- To estimate these, we entrust to the `R`-package `mclust`.

4.3 Fitting a mixture

- We fit the model

```
library(mclust)
fit <- Mclust(ln_Error_corrected, 2 , "E")# 2 clusters; "E"qual variances
pr <- fit$parameters$pro[1]
pr
```

```
## [1] 0.728314
```

- The chance of coming from lot 1 is around 73%.

```
means <- fit$parameters$mean
means
```

```
##           1           2 
## -0.05174452  0.05406515
```

- The mean in lot 1 is around -5.2%
- The mean in lot 2 is around 5.4%

```
sigma <- sqrt(fit$parameters$variance$sigmasq)
sigma
```

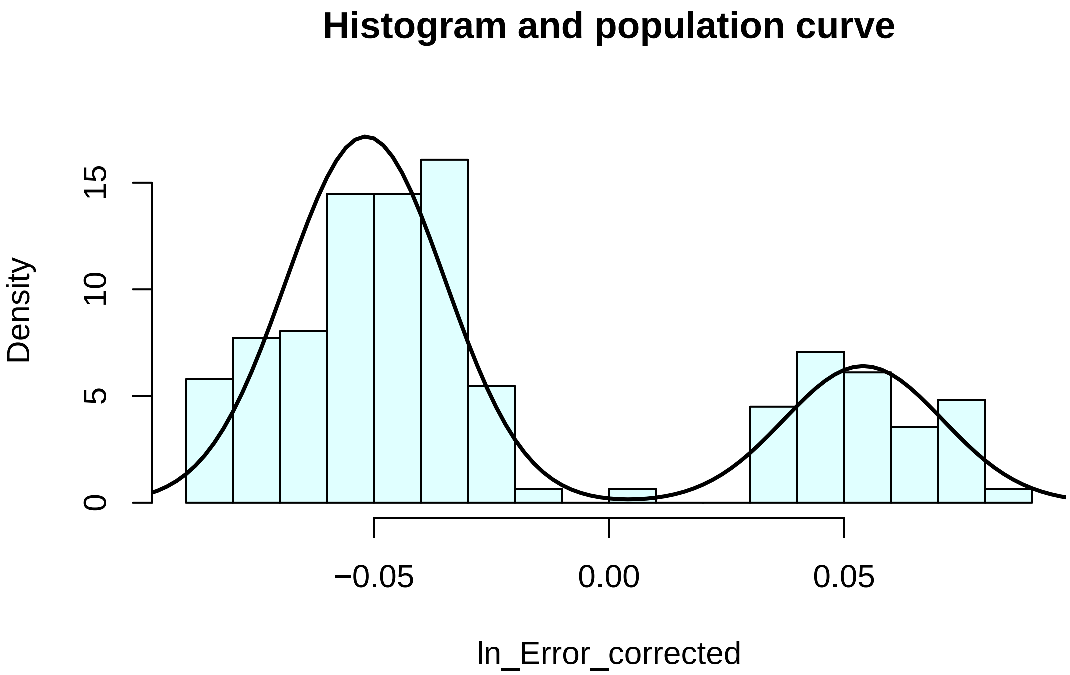
```
## [1] 0.01692654
```

- σ is around 1.7%

4.4 Comparing model and data

- We compare the histogram with the fitted normal curves.

```
hist(ln_Error_corrected,breaks=15,col="lightcyan",probability = TRUE,ylim=c(0,18),main="Histogram and po
curve(pr*dnorm(x,means[1],sigma)+(1-pr)*dnorm(x,means[2],sigma),-.1,.1,add=TRUE,lwd=2)
```



4.5 Concluding remarks

- Estimate of σ was 1.7%. In relation to the 220 nF lot we estimated 2.0%, which is comparable.
  - 3 sigma limits for the correct lot 1 values:

$$-5.2\% \pm 3 * 1.7\% = [-10.3; -0.1]\%$$

  - 3 sigma limits for the correct lot 2 values:

$$5.4\% \pm 3 * 1.7\% = [0.3; 10.5]\%$$

- The lots do not completely respect the tolerance of 10%. However, in the sample the minimum is -8.9% and the maximum 8.3%.

- The difference in lot means is $5.4\% - (-5, 2)\% = 10.6$.
- This indicates that the variation between lots is much greater than the variation within lots.
- This is also clearly illustrated by the histogram/density plots.