

# ASTA

The ASTA team

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# 1 Software

## 1.1 Rstudio

- Make a folder on your computer where you want to keep files to use in **Rstudio**. **Do NOT use Danish characters æ, ø, å** in the folder name (or anywhere in the path to the folder).
- Set the working directory to this folder: **Session -> Set Working Directory -> Choose Directory** (shortcut: Ctrl+Shift+H).
- Make the change permanent by setting the default directory in: **Tools -> Global Options -> Choose Directory**.

## 1.2 R extensions

- The functionality of **R** can be extended through libraries or packages (much like plugins in browsers etc.). Some are installed by default in **R** and you just need to load them.
- To install a new package in **Rstudio** use the menu: **Tools -> Install Packages**
- You need to know the name of the package you want to install. You can also do it through a command:

```
install.packages("mosaic")
```

- When it is installed you can load it through the **library** command:

```
library(mosaic)
```

- This loads the **mosaic** package which has a lot of convenient functions for this course (we will get back to that later). It also prints a lot of info about functions that have been changed by the **mosaic** package, but you can safely ignore that.

## 1.3 R help

- You get help via `?<command>`:

```
?sum
```

- Use **tab** to make **Rstudio** guess what you have started typing.
- Search for help:

```
help.search("plot")
```

- You can find a cheat sheet with the **R** functions we use for this course [here](#).

## 2 Data

### 2.1 Data example

We use data about penguins from the R package palmerpenguins

```
pingviner <- palmerpenguins::penguins
pingviner

## # A tibble: 344 x 8
##   species island   bill_length_mm bill_depth_mm flipper_length_mm body_mass_g
##   <fct>   <fct>         <dbl>         <dbl>         <int>         <int>
## 1 Adelie  Torgersen         39.1          18.7           181           3750
## 2 Adelie  Torgersen         39.5          17.4           186           3800
## 3 Adelie  Torgersen         40.3           18            195           3250
## 4 Adelie  Torgersen         NA             NA             NA             NA
## 5 Adelie  Torgersen         36.7          19.3           193           3450
## 6 Adelie  Torgersen         39.3          20.6           190           3650
## 7 Adelie  Torgersen         38.9          17.8           181           3625
## 8 Adelie  Torgersen         39.2          19.6           195           4675
## 9 Adelie  Torgersen         34.1          18.1           193           3475
## 10 Adelie Torgersen         42            20.2           190           4250
## # i 334 more rows
## # i 2 more variables: sex <fct>, year <int>
```

- What is fundamentally different about the the variables (columns) `species` and `body_mass_g`?

### 2.2 Data types

#### 2.2.1 Quantitative variables

- The measurements have numerical values.
- Quantative data often comes about in one of the following ways:
  - **Continuous variables**: measurements of time, length, size, age, mass, etc.
  - **Discrete variables**: counts of e.g. words in a text, hits on a webpage, number of arrivals to a queue in one hour, etc.
- Measurements like this have a well-defined scale and in **R** they are stored as the type **numeric**.
- It is important to be able to distinguish between discrete count variables and continuous variables, since this often determines how we describe the uncertainty of a measurement.
- Are any of the measurements in our data set quantitative?

---

#### 2.2.2 Categorical/qualitative variables

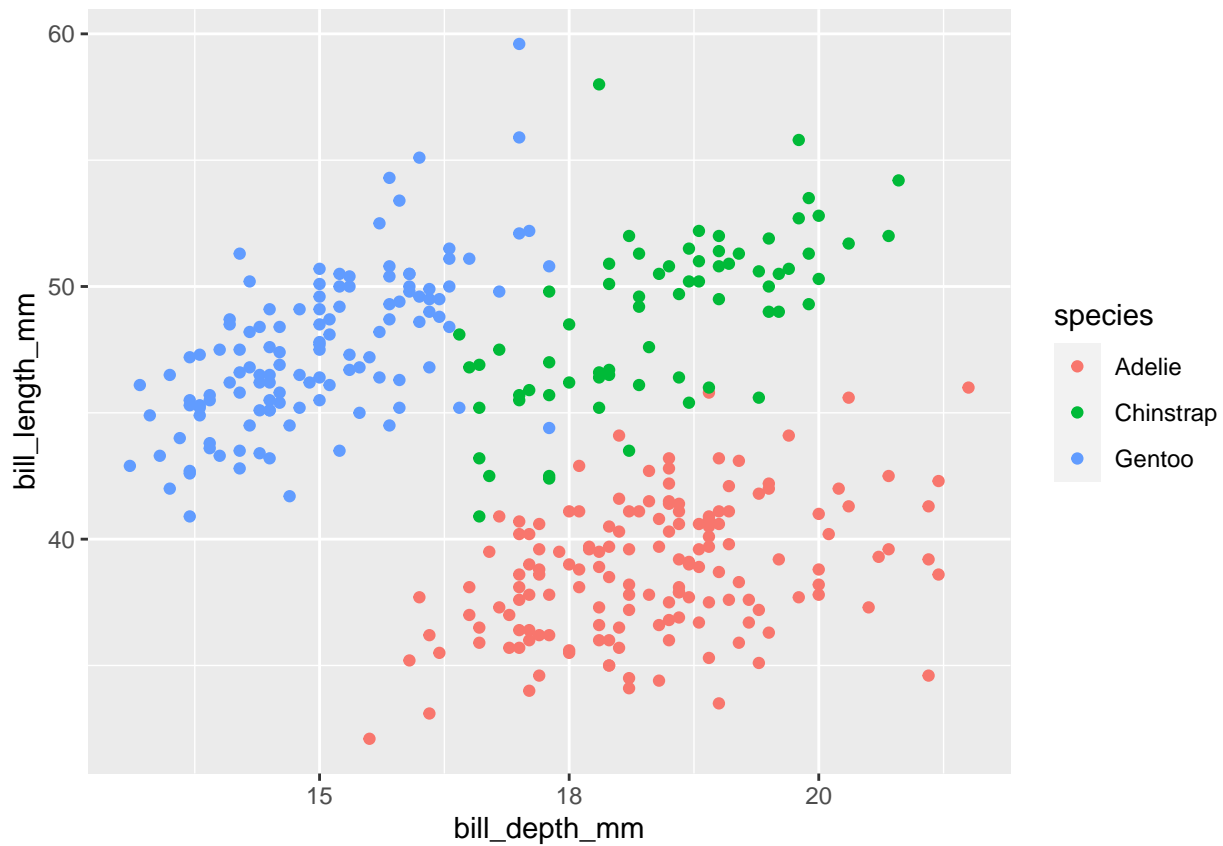
- The measurement is one of a set of given categories, e.g. sex (male/female), social status, satisfaction score (low/medium/high), etc.
- The measurement is usually stored (which is also recommended) as a **factor** in **R**. The possible categories are called **levels**. Example: the levels of the factor “sex” is male/female.
- Factors have two so-called scales:
  - **Nominal scale**: There is no natural ordering of the factor levels, e.g. sex and hair color.
  - **Ordinal scale**: There is a natural ordering of the factor levels, e.g. social status and satisfaction score. A factor in **R** can have a so-called **attribute** assigned, which tells if it is ordinal.
- Are any of the measurements in our data set categorical/qualitative?

### 3 Graphics for quantitative variables

#### 3.1 Scatterplot

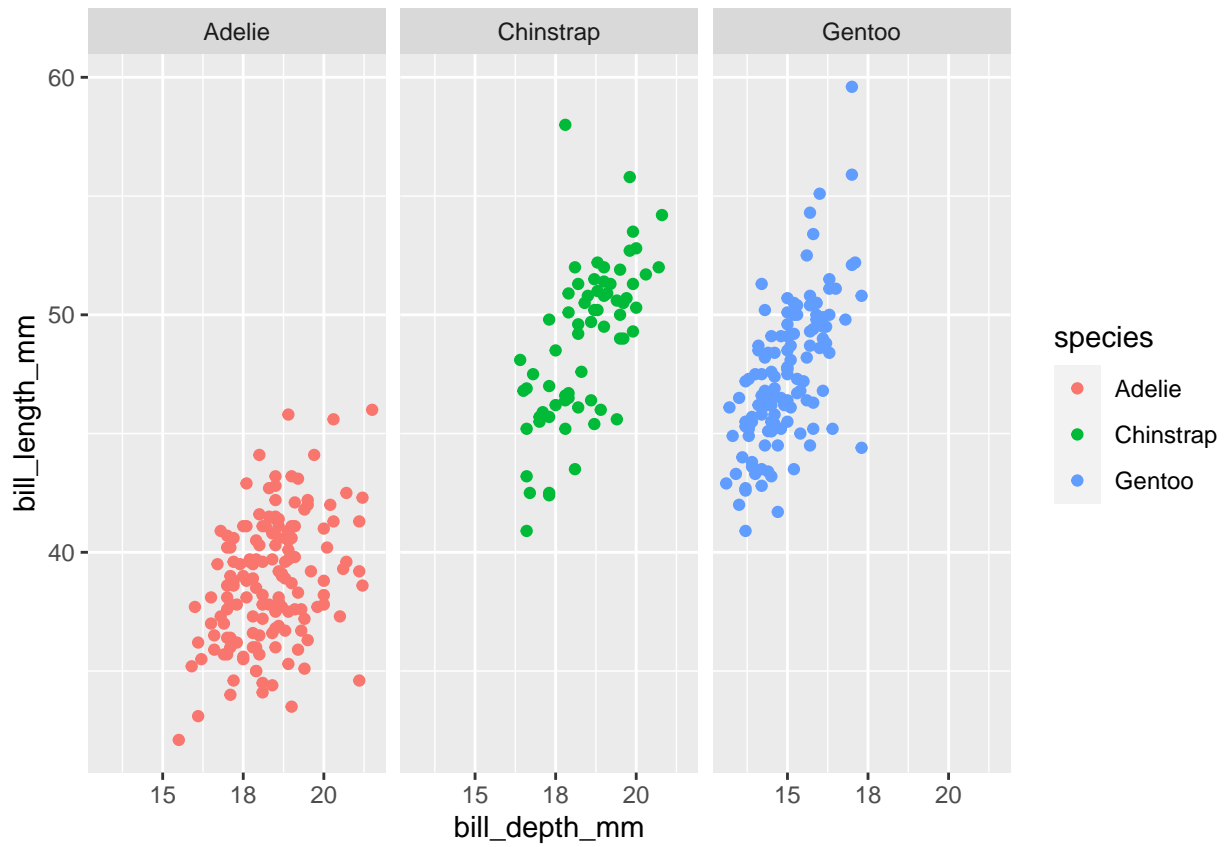
- To study the relation between two quantitative variables a scatterplot is used:

```
gf_point(bill_length_mm ~ bill_depth_mm, color = ~ species, data = pingviner)
```



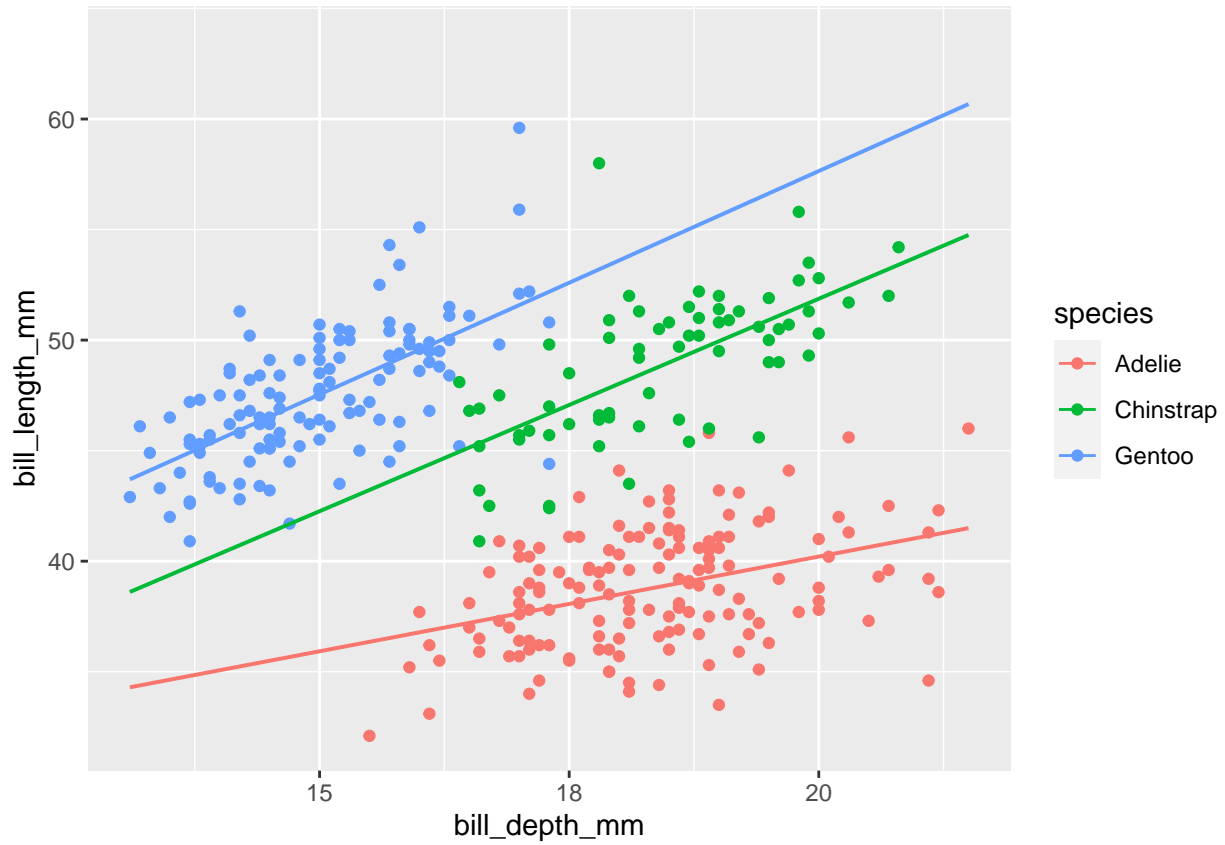
- We could also draw the graph for each species:

```
gf_point(bill_length_mm ~ bill_depth_mm | species, color = ~ species, data = pingviner)
```



- If we want a regression line along with the points we can do:

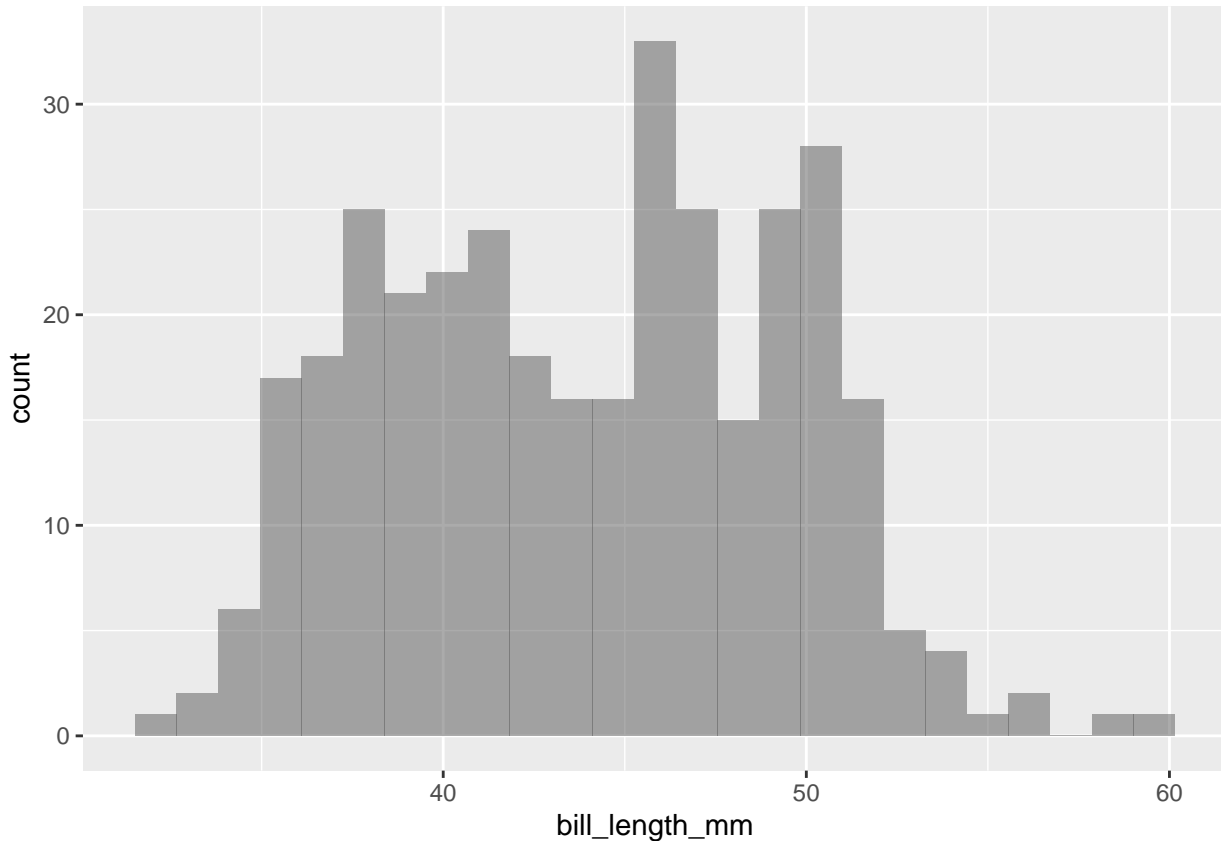
```
gf_point(bill_length_mm ~ bill_depth_mm, color = ~ species, data = pingviner) %>%  
gf_lm()
```



### 3.2 Histogram

- For a single quantitative variable a histogram offers more details:

```
gf_histogram(~ bill_length_mm, data = pingviner)
```



- How to make a histogram for some variable  $x$ :
  - Divide the interval from the minimum value of  $x$  to the maximum value of  $x$  in an appropriate number of equal sized sub-intervals.
  - Draw a box over each sub-interval with the height being proportional to the number of observations in the sub-interval.

## 4 Summaries of quantitative variables

### 4.1 Percentiles

- The  $p$ th percentile is a value such that at least  $p\%$  of the sample lies below or at this value and at least  $(100 - p)\%$  of the sample lies above or at the value.

```
Q <- quantile(bill_length_mm ~ species, data = pingviner, na.rm = TRUE)
Q
```

```
##   species 0% 25% 50% 75% 100%
## 1  Adelie 32 37 39 41 46
## 2 Chinstrap 41 46 50 51 58
## 3  Gentoo 41 45 47 50 60
```

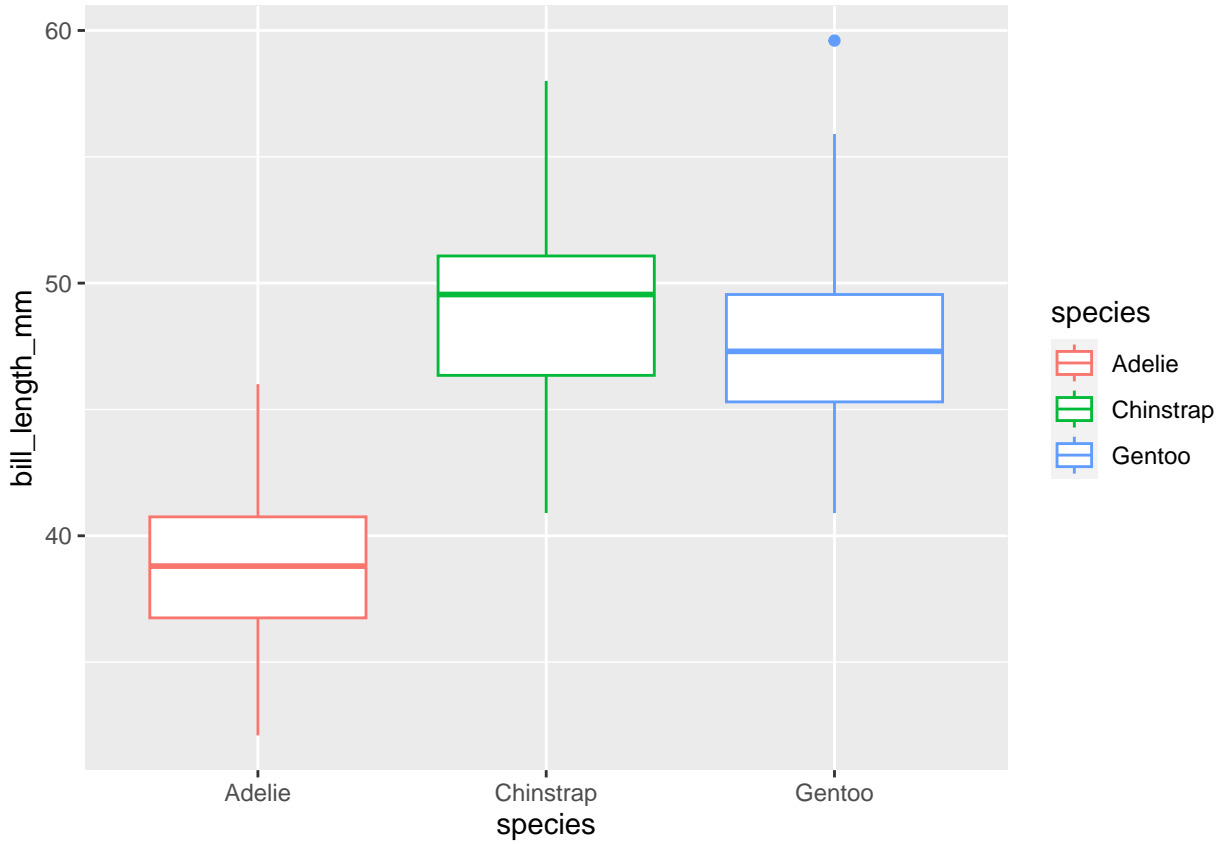
- 50-percentile is the **median** and it is a measure of the center of data as the number of data points below the median is the same as the number above the median.
- 0-percentile is the **minimum** value.
- 25-percentile is called the **lower quartile** (Q1). Median of lower 50% of data.
- 75-percentile is called the **upper quartile** (Q3). Median of upper 50% of data.
- 100-percentile is the **maximum** value.

- **Interquartile Range (IQR)**: a measure of variability given by the difference of the upper and lower quartiles.

## 4.2 Boxplot

Boxplot can be good for comparing groups (notice we put the values on the y-axis here as it is more conventional for boxplots):

```
gf_boxplot(bill_length_mm ~ species, color = ~ species, data = pingviner)
```



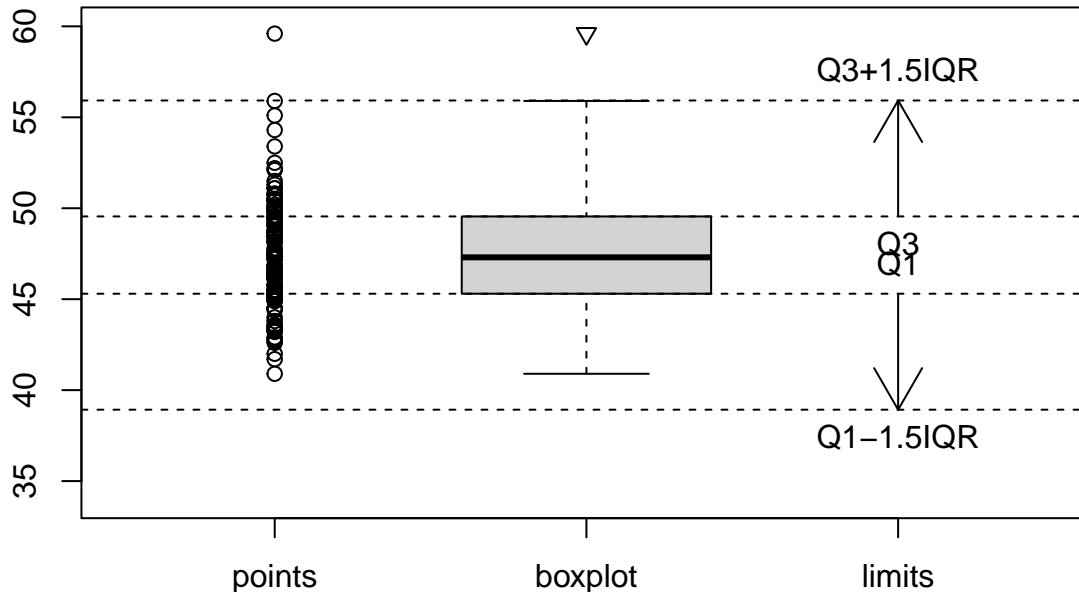
### 4.2.1 How to draw a box plot

- Box:
  - Calculate the median, lower and upper quartiles.
  - Plot a line by the median and draw a box between the upper and lower quartiles.
- Whiskers:
  - Calculate interquartile range and call it IQR.
  - Calculate the following values:
    - \*  $L = \text{lower quartile} - 1.5 \cdot \text{IQR}$
    - \*  $U = \text{upper quartile} + 1.5 \cdot \text{IQR}$
  - Draw a line from lower quartile to the smallest measurement, which is larger than  $L$ .
  - Similarly, draw a line from upper quartile to the largest measurement which is smaller than  $U$ .
- Outliers: Measurements smaller than  $L$  or larger than  $U$  are drawn as circles.

*Note: Whiskers are minimum and maximum of the observations that are not deemed to be outliers.*



## Gentoo bill length



### 4.3 Measures of center of data: Mean and median

- A number of numerical summaries can be retrieved using the `favstats` command:

```
favstats(bill_length_mm ~ species, data = pingviner)
```

```
##   species min Q1 median Q3 max mean sd  n missing
## 1  Adelie  32 37   39 41  46  39 2.7 151      1
## 2 Chinstrap 41 46   50 51  58  49 3.3  68      0
## 3  Gentoo  41 45   47 50  60  48 3.1 123      1
```

- The observed values of `bill_length_mm` are  $y_1 = 46.1, y_2 = 50, \dots, y_n = 49.9$ , where there are a total of  $n = 123$  values.

As previously defined this constitutes a **sample**.

- mean** = 48 is the **average** of the sample, which is calculated by

$$\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i.$$

We may also call  $\bar{y}$  the **(empirical) mean** or the **sample mean**. It is calculated using `mean()` in **R**.

- median** = 47 is calculated using `median()` in **R**.
- An important property of the **mean** and the **median** is that they have the same unit as the observations (e.g. millimeter).

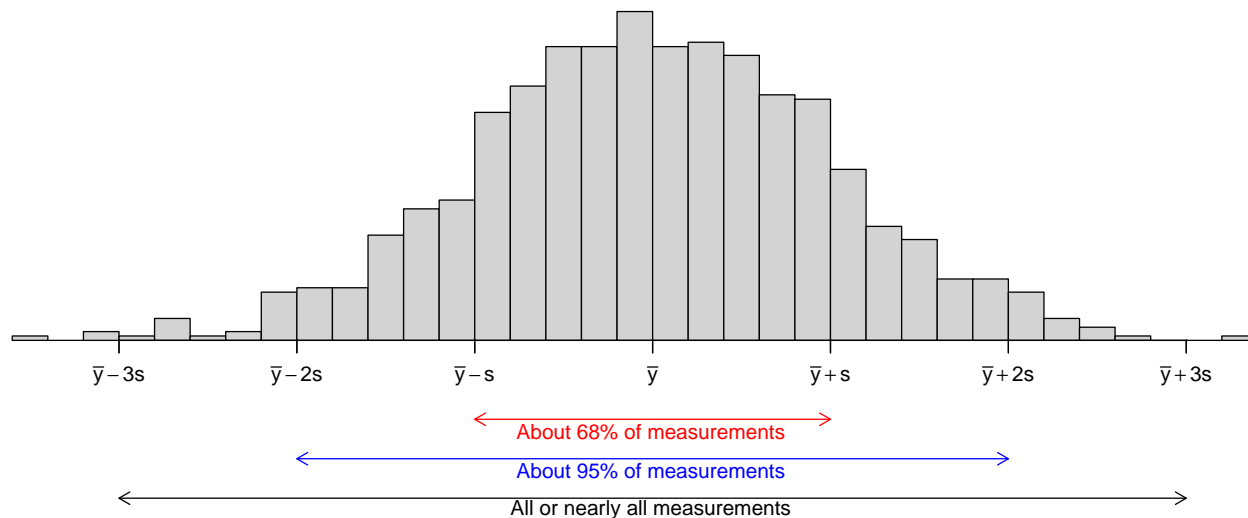
### 4.4 Measures of variability of data: range, standard deviation and variance

- The **range** is the difference of the largest and smallest observation (`range()` in **R**).
- The **(empirical) variance** (`var()` in **R**) is the average of the squared deviations from the mean:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2.$$

- **sd = standard deviation** =  $s = \sqrt{s^2}$  (**sd()** in **R**).
- Note: If the observations are measured in mm, the **variance** has unit  $\text{mm}^2$  which is hard to interpret. The **standard deviation** on the other hand has the same unit as the observations.
- The standard deviation describes how much data varies around the (empirical) mean.

#### 4.4.1 The empirical rule



If the histogram of the sample looks like a bell shaped curve, then

- about 68% of the observations lie between  $\bar{y} - s$  and  $\bar{y} + s$ .
- about 95% of the observations lie between  $\bar{y} - 2s$  and  $\bar{y} + 2s$ .
- All or almost all (99.7%) of the observations lie between  $\bar{y} - 3s$  and  $\bar{y} + 3s$ .

## 5 Probability of events

### 5.1 The concept of probability

- Experiment: Measure the waiting times in a queue where we note 1, if it exceeds 2 minutes and 0 otherwise.
- The experiment is carried out  $n$  times with results  $y_1, y_2, \dots, y_n$ . There is **random variation** in the outcome, i.e. sometimes we get a 1 other times a 0.
- **Empirical probability** of exceeding 2 minutes:

$$p_n = \frac{\sum_{i=1}^n y_i}{n}.$$

- **Theoretical probability** of exceeding 2 minutes:

$$\pi = \lim_{n \rightarrow \infty} p_n.$$

- We try to make inference about  $\pi$  based on a sample, e.g. “Is  $\pi > 0.1$ ?” (“do more than 10% of the customers experience a waiting time in excess of 2 minutes?”).
- Statistical inference is concerned with such questions when we only have a finite sample.

### 5.2 Actual experiment

- On February 23, 2017, a group of students were asked how long time (in minutes) they waited in line last time they went to the canteen at AAU’s Copenhagen campus:

```
y_canteen <- c(2, 5, 1, 6, 1, 1, 1, 1, 3, 4, 1, 2, 1, 2, 2, 2, 4, 2, 2, 5, 20, 2, 1, 1, 1, 1)
x_canteen <- ifelse(y_canteen > 2, 1, 0)
x_canteen
```

```
## [1] 0 1 0 1 0 0 0 0 1 1 0 0 0 0 0 0 1 0 0 1 1 0 0 0 0 0
```

- Empirical probability of waiting more than 2 minutes:

```
p_canteen <- sum(x_canteen) / length(x_canteen)
p_canteen
```

```
## [1] 0.27
```

- Question: Is the population probability  $\pi > 1/3$ ?
- Notice: One student said he had waited for 20 minutes (we doubt that; he was trying to make himself interesting. Could consider ignoring that observation).

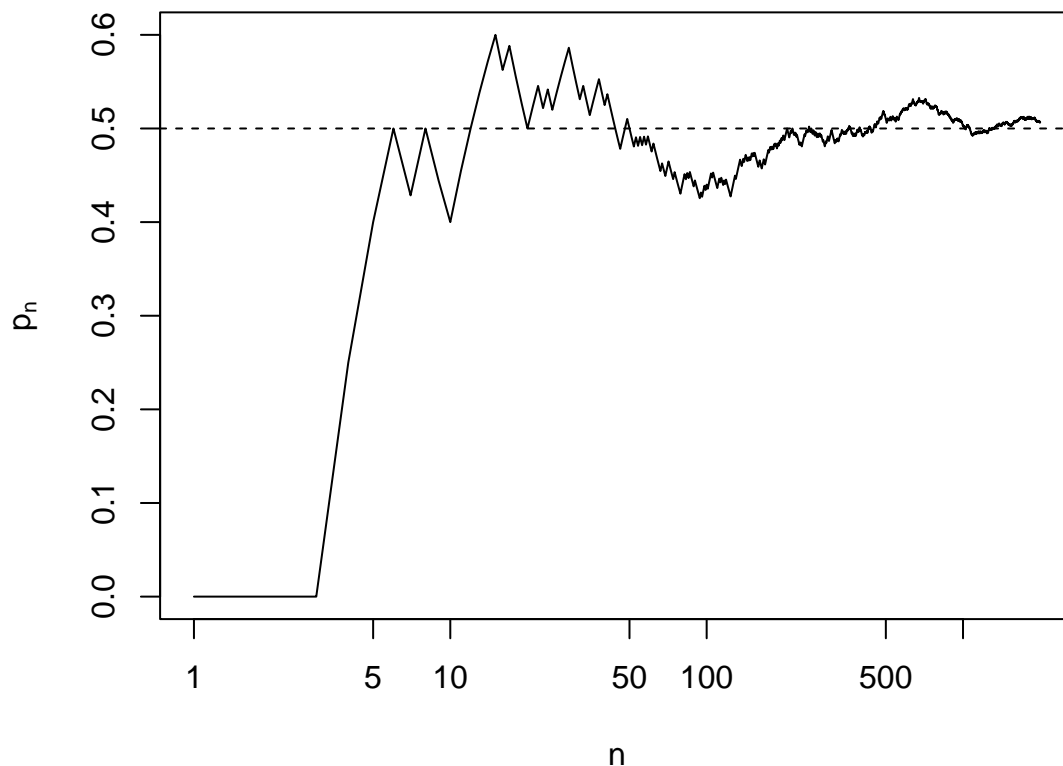
### 5.3 Another experiment

- John Kerrich, a South African mathematician, was visiting Copenhagen when World War II broke out. Two days before he was scheduled to fly to England, the Germans invaded Denmark. Kerrich spent the rest of the war interned at a camp in Hald Ege near Viborg, Jutland, and to pass the time he carried out a series of experiments in probability theory. In one, he tossed a coin 10,000 times. His results are shown in the following graph.
- Below,  $x$  is a vector with the first 2,000 outcomes of John Kerrich's experiment (0 = tail, 1 = head):

```
head(x, 10)
```

```
## [1] 0 0 0 1 1 1 0 1 0 0
```

- Plot of the empirical probability  $p_n$  of getting a head against the number of tosses  $n$ :



(The horizontal axis is on a log scale).

## 5.4 Definitions

- **Sample space:** All possible outcomes of the experiment.
- **Event:** A subset of the sample space.

We conduct the experiment  $n$  times. Let  $\#(A)$  denote how many times we observe the event  $A$ .

- **Empirical probability** of the event  $A$ :

$$p_n(A) = \frac{\#(A)}{n}.$$

- **Theoretical probability** of the event  $A$ :

$$P(A) = \lim_{n \rightarrow \infty} p_n(A)$$

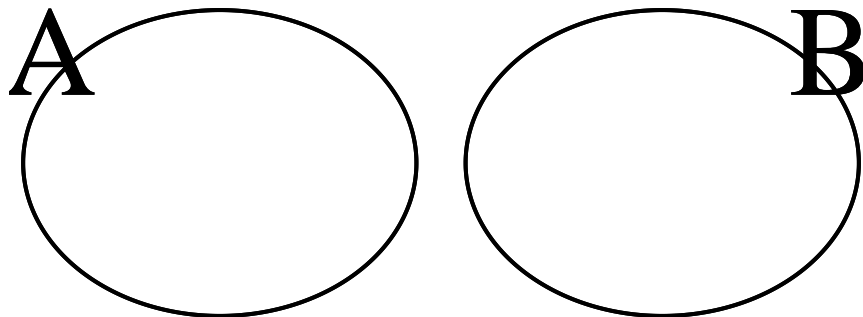
- We always have  $0 \leq P(A) \leq 1$ .

- **Examples:**

- **Example 1** Tossing a coin once. The sample space is  $S = \{H, T\}$ .  $E = \{H\}$  is an event.
- **Example 2** Tossing a die. The sample space is  $S = \{1, 2, 3, 4, 5, 6\}$ .  $E = \{2, 4, 6\}$  is an event, which can be described in words as **the number is even**.
- **Example 3** Tossing a coin twice. The sample space is  $S = \{HH, HT, TH, TT\}$ .  $E = \{HH, HT\}$  is an event, which can be described in words as **the first toss results in a Heads**.
- **Example 4** Tossing a die twice. The sample space is  $S = \{(i, j) : i, j = 1, 2, \dots, 6\}$ , which contains 36 outcomes. **The sum of the results of the two tosses is equal to 10** is an event.

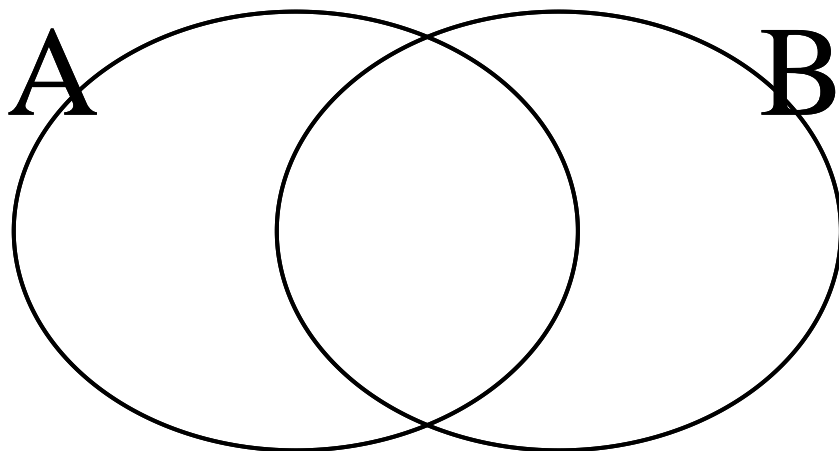
## 5.5 Theoretical probabilities of two events

- If the two events  $A$  and  $B$  are **disjoint** (non-overlapping) then
  - $\#(A \text{ and } B) = 0$  implying that  $P(A \text{ and } B) = 0$ .
  - $\#(A \text{ or } B) = \#(A) + \#(B)$  implying that  $P(A \text{ or } B) = P(A) + P(B)$ .



- If the two events  $A$  and  $B$  are **not disjoint** then the more general formula is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$



## 5.6 Conditional probability

- Say we consider two events  $A$  and  $B$ . Then the **conditional probability** of  $A$  given (or conditional on) the event  $B$  is written  $P(A | B)$  and is defined by

$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)}.$$

- The above probability can be understood as: “how probable  $A$  is if we know that  $B$  has happened”.

### 5.6.1 Example: 3 coin tosses

- **Random/stochastic variable:** A function  $Y$  that translates an outcome of the experiment into a number.
- Possible outcomes in an experiment with 3 coin tosses:
  - 0 heads (TTT)
  - 1 head (HTT, THT, TTH)
  - 2 heads (HHT, HTH, THH)
  - 3 heads (HHH)
- The above events are disjoint and make up the whole sample space.
- Let  $Y$  be the number of heads in the experiment:  $Y(TTT) = 0, Y(HTT) = 1, \dots$
- Assume that each outcome is equally likely, i.e. probability  $1/8$  for each event. Then,
  - $P(\text{no heads}) = P(Y = 0) = P(TTT) = 1/8$ .
  - $P(\text{one head}) = P(Y = 1) = P(HTT \text{ or } THT \text{ or } TTH) = P(HTT) + P(THT) + P(TTH) = 3/8$ .
  - Similarly for 2 or 3 heads.
- So, the distribution of  $Y$  is

Number of heads, $Y$	0	1	2	3
Probability	$1/8$	$3/8$	$3/8$	$1/8$

## 6 Distribution of general random variables

### 6.1 Probability distribution

- We are conducting an experiment where we make a quantitative measurement  $Y$  (a random variable), e.g. the number of words in an ad or the waiting time in a queue.

- In advance there are many possible outcomes of the experiment, i.e.  $Y$ 's value has an uncertainty, which we quantify by the **probability distribution** of  $Y$ .
- For any interval  $(a, b)$ , the distribution states the probability of observing a value of the random variable  $Y$  in this interval:

$$P(a < Y < b), \quad -\infty < a < b < \infty.$$

- $Y$  is **discrete** if we can enumerate all the possible values of  $Y$ , e.g. the number of words in an ad.
- $Y$  is **continuous** if  $Y$  can take any value in a interval, e.g. a measurement of waiting time in a queue.

### 6.1.1 Sample

We conduct an experiment  $n$  times, where the outcome of the  $i$ th experiment corresponds to a measurement of a random variable  $Y_i$ , where we assume

- The experiments are **independent**
- The variables  $Y_1, \dots, Y_n$  have the **same distribution**

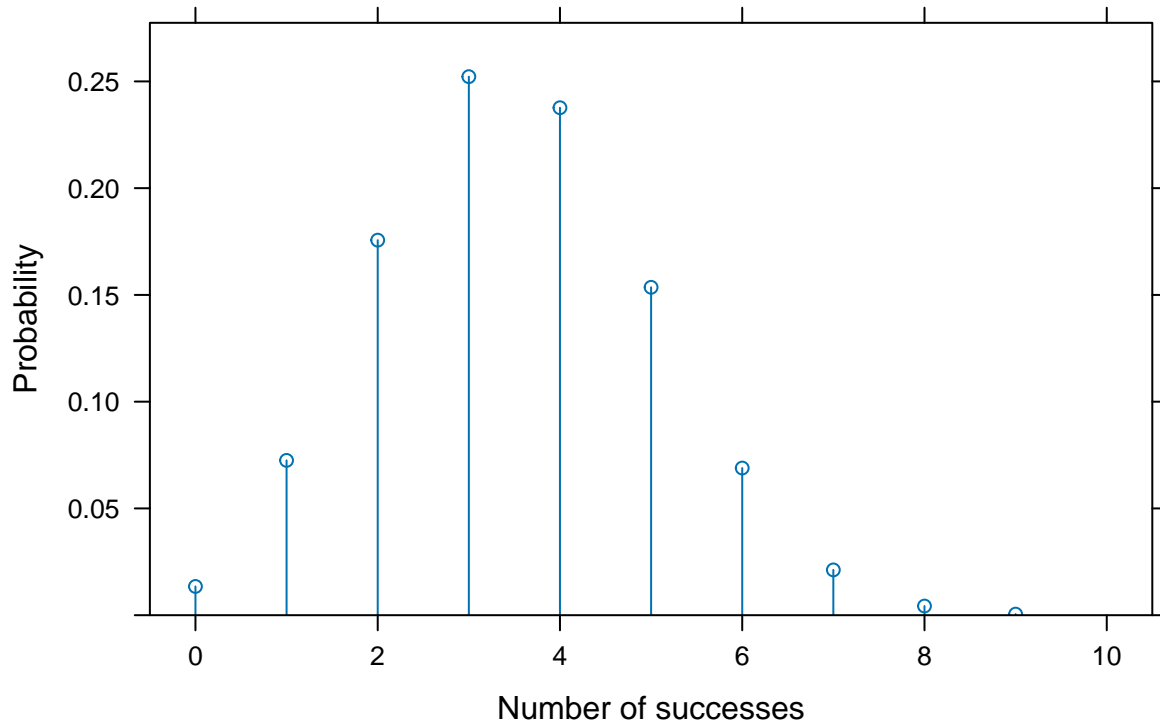
## 6.2 Population parameters

- When the sample size grows, then e.g. the mean of the sample,  $\bar{y}$ , will stabilize around a fixed value,  $\mu$ , which is usually unknown. The value  $\mu$  is called the **population mean**.
- Correspondingly, the standard deviation of the sample,  $s$ , will stabilize around a fixed value,  $\sigma$ , which is usually unknown. The value  $\sigma$  is called the **population standard deviation**.
- Notation:
  - $\mu$  (mu) denotes the population mean.
  - $\sigma$  (sigma) denotes the population standard deviation.

Population	Sample
$\mu$	$\bar{y}$
$\sigma$	$s$

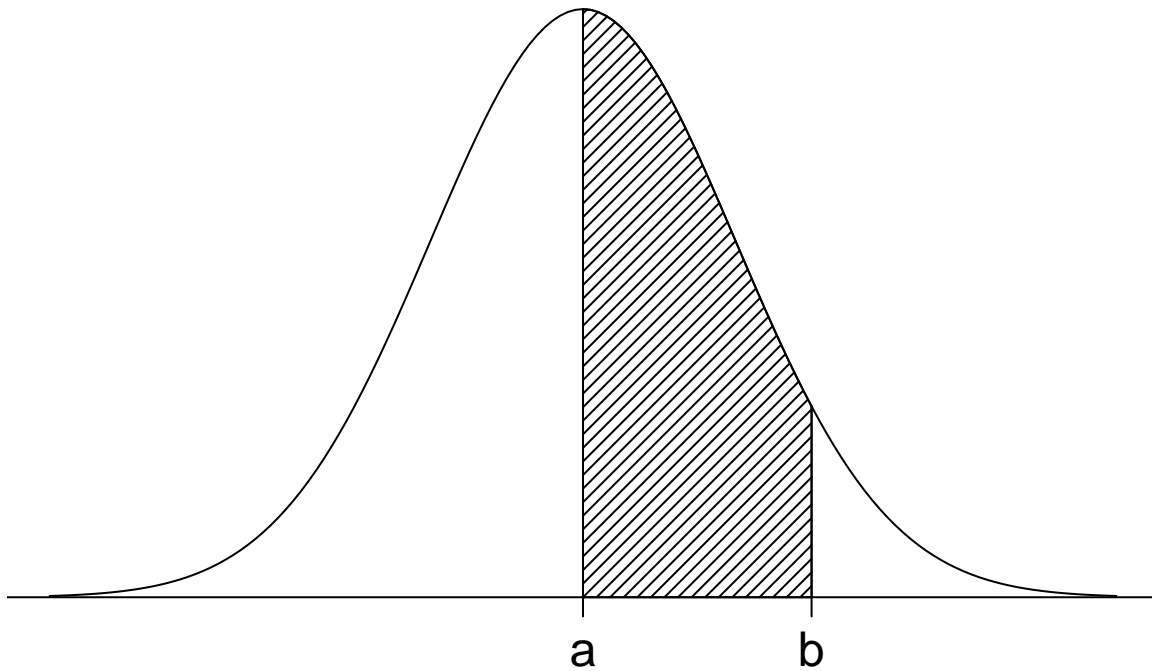
```
# The binomial distribution with n = 10 and pi = 0.35:
plotDist("binom", size = 10, prob = 0.35, xlim = c(-0.5, 10.5),
        ylab = "Probability", xlab = "Number of successes", main = "binom(n = 10, prob = 0.35)")
```

### binom( $n = 10$ , prob = 0.35)



### 6.3 Distribution of a continuous random variable

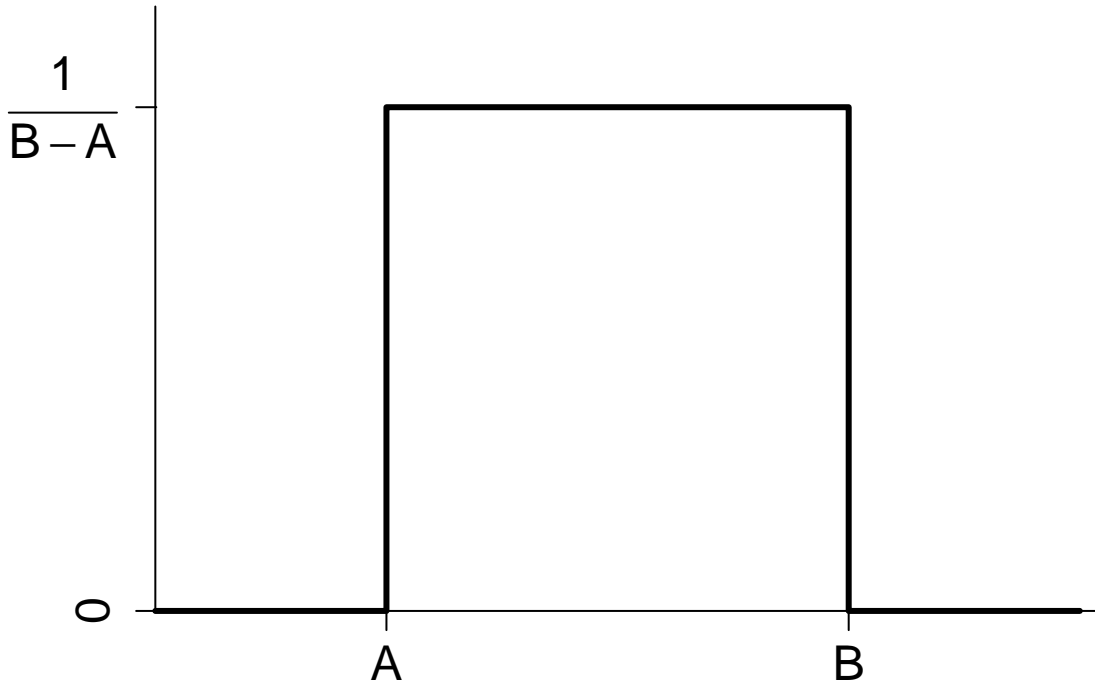
- The distribution of a continuous random variable  $Y$  is characterized by the so-called probability density function  $f_Y$ .



- The area under the graph of the probability density function between  $a$  and  $b$  is equal to the probability of an observation in this interval.

- $f_Y(y) \geq 0$  for all real numbers  $y$ .
- The area under the graph for  $f_Y$  is equal to 1.
- For example the **uniform distribution** from  $A$  to  $B$ :

$$f_Y(y) = \begin{cases} \frac{1}{B-A} & A \leq y \leq B \\ 0 & \text{otherwise} \end{cases}$$

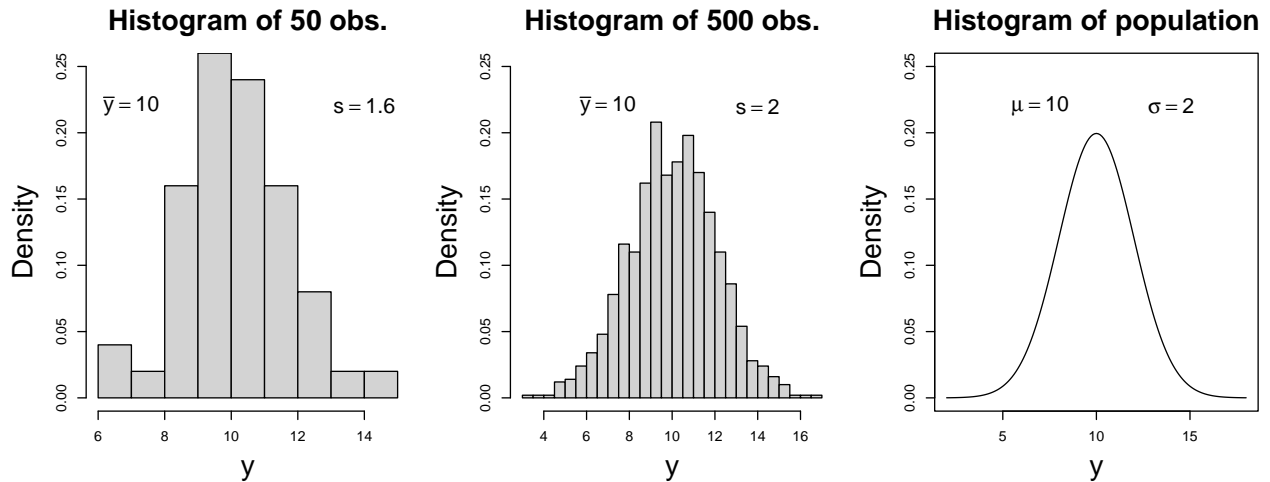


## 6.4 Density function

### 6.4.1 Increasing number of observations

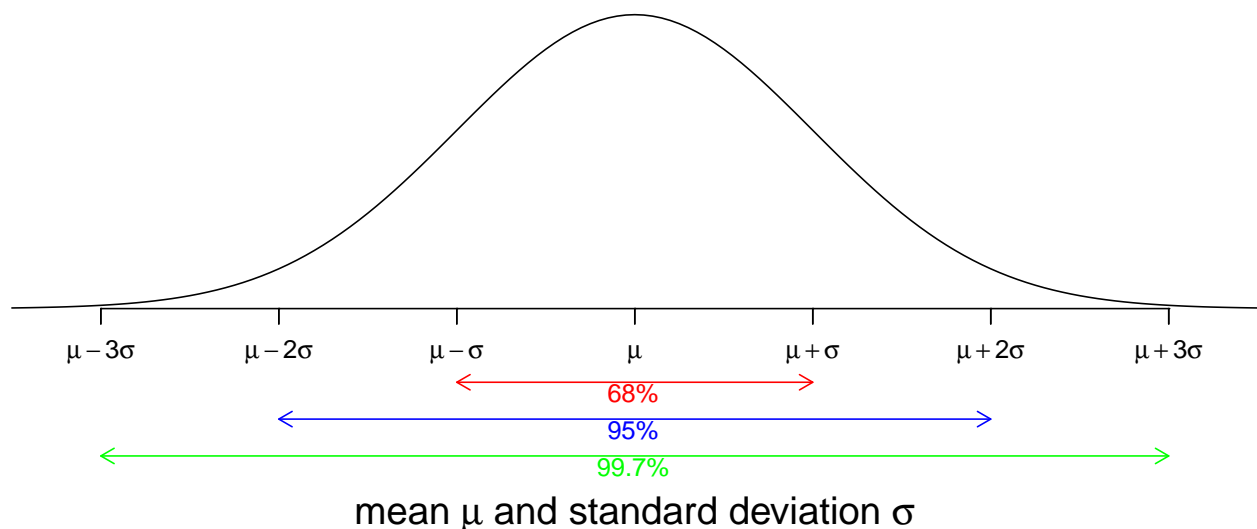
- Another way to think about the density is in terms of the histogram.
- If we draw a histogram for a sample where the area of each box corresponds to the relative frequency of each interval, then the total area will be 1.
- When the number of observations (sample size) increase we can make a finer interval division and get a more smooth histogram.
- We can imagine an infinite number of observations, which would produce a nice smooth curve, where the area below the curve is 1. A function derived this way is also what we call the **probability density function**.





#### 6.4.2 Reach of the normal distribution

### Density of the normal distribution



Interpretation of standard deviation:

- $\approx 68\%$  of the population is within 1 standard deviation of the mean.
- $\approx 95\%$  of the population is within 2 standard deviations of the mean.
- $\approx 99.7\%$  of the population is within 3 standard deviations of the mean.

#### 6.4.3 Normal $z$ -score

- If  $Y \sim \text{norm}(\mu, \sigma)$  then the corresponding so-called  $z$ -score is

$$Z = \frac{Y - \mu}{\sigma} = \frac{\text{observation} - \text{mean}}{\text{standard deviation}}$$

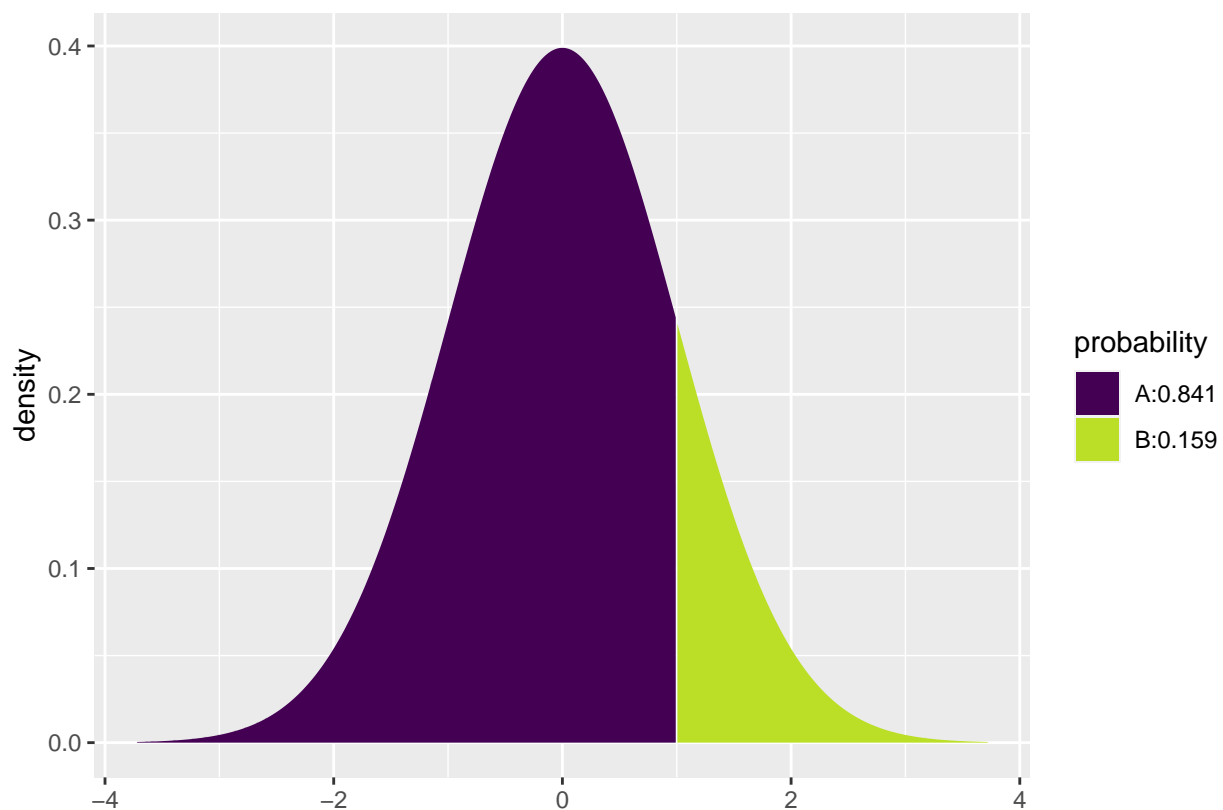
- I.e.  $Z$  counts the number of standard deviations that the observation lies away from the mean, where a negative value tells that we are below the mean.
- We have that  $Z \sim \text{norm}(0, 1)$ , i.e.  $Z$  has zero mean and standard deviation one.
- This implies that

- $Z$  lies between  $-1$  and  $1$  with probability 68%
- $Z$  lies between  $-2$  and  $2$  with probability 95%
- $Z$  lies between  $-3$  and  $3$  with probability 99.7%
- It also implies that:
  - The probability of  $Y$  being between  $\mu - z\sigma$  and  $\mu + z\sigma$  is equal to the probability of  $Z$  being between  $-z$  and  $z$ .

#### 6.4.4 Calculating probabilities in the standard normal distribution

- The function `pnorm` always outputs the area to the left of the  $z$ -value (quantile/percentile) we give as input (variable `q` in the function), i.e. it outputs the probability of getting a value less than  $z$ . The first argument of `pnorm` denotes the distribution we are considering.

```
# For a standard normal distribution the probability of getting a value less than 1 is:
left_prob <- pnorm("norm", q = 1, mean = 0, sd = 1)
```



```
left_prob
```

```
## [1] 0.84
```

So  $q=1$  corresponds to the 0.84-percentile/quantile for the standard normal distribution

- Here there is a conflict between **R** and the textbook, since the book always considers right tail probabilities. Since the total area is 1 and we have the left probability we easily get the right probability:

```
right_prob <- 1 - left_prob
right_prob
```

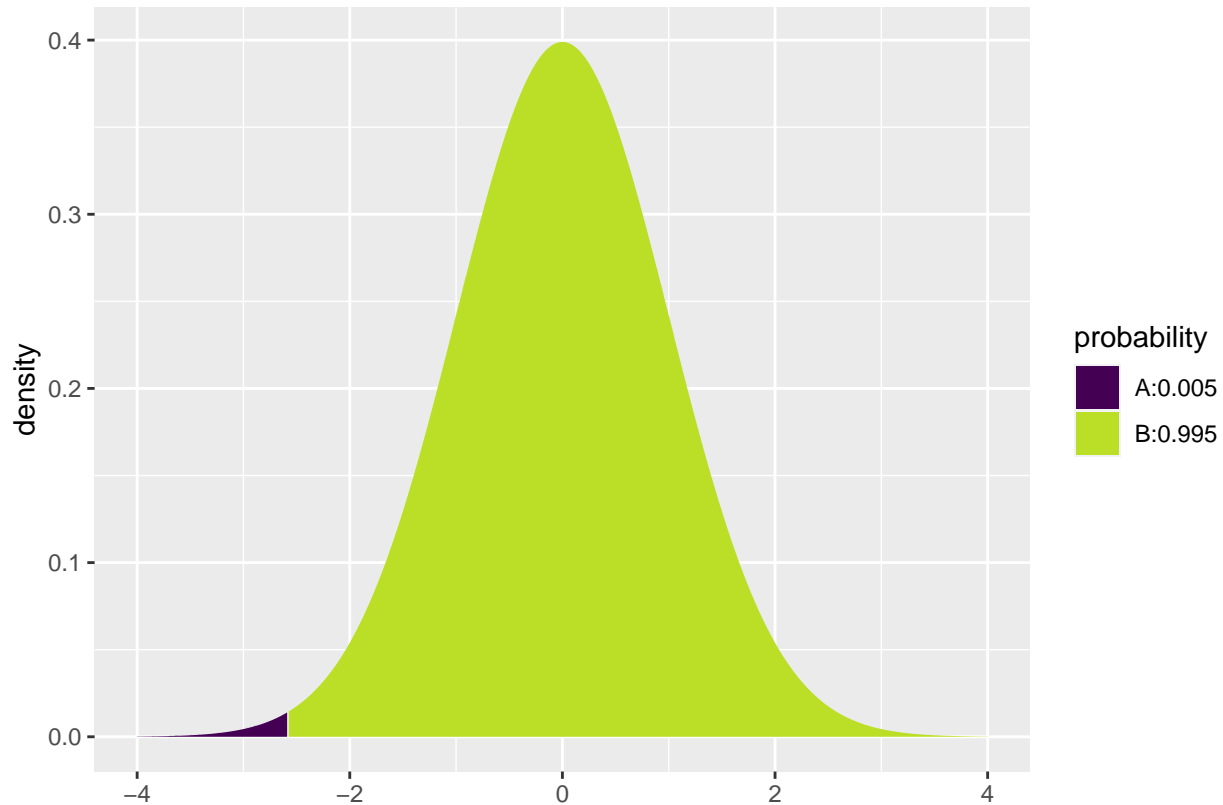
```
## [1] 0.16
```

- For  $z = 1$  we have a right probability of  $p = 0.1587$ , so the probability of an observation between  $-1$  and  $1$  is equal to  $1 - 2 \cdot 0.1587 = 0.6826 = 68.26\%$  due to symmetry.

#### 6.4.5 Calculating $z$ -values (quantiles) in the standard normal distribution

- If we have a probability and want to find the corresponding  $z$ -value we again need to decide on left/right probability. The default in **R** is to find the left probability, so if we want the  $z$ -value with e.g. 0.5% probability to the left we get:

```
left_z <- qdist("norm", p = 0.005, mean = 0, sd = 1, xlim = c(-4, 4))
```

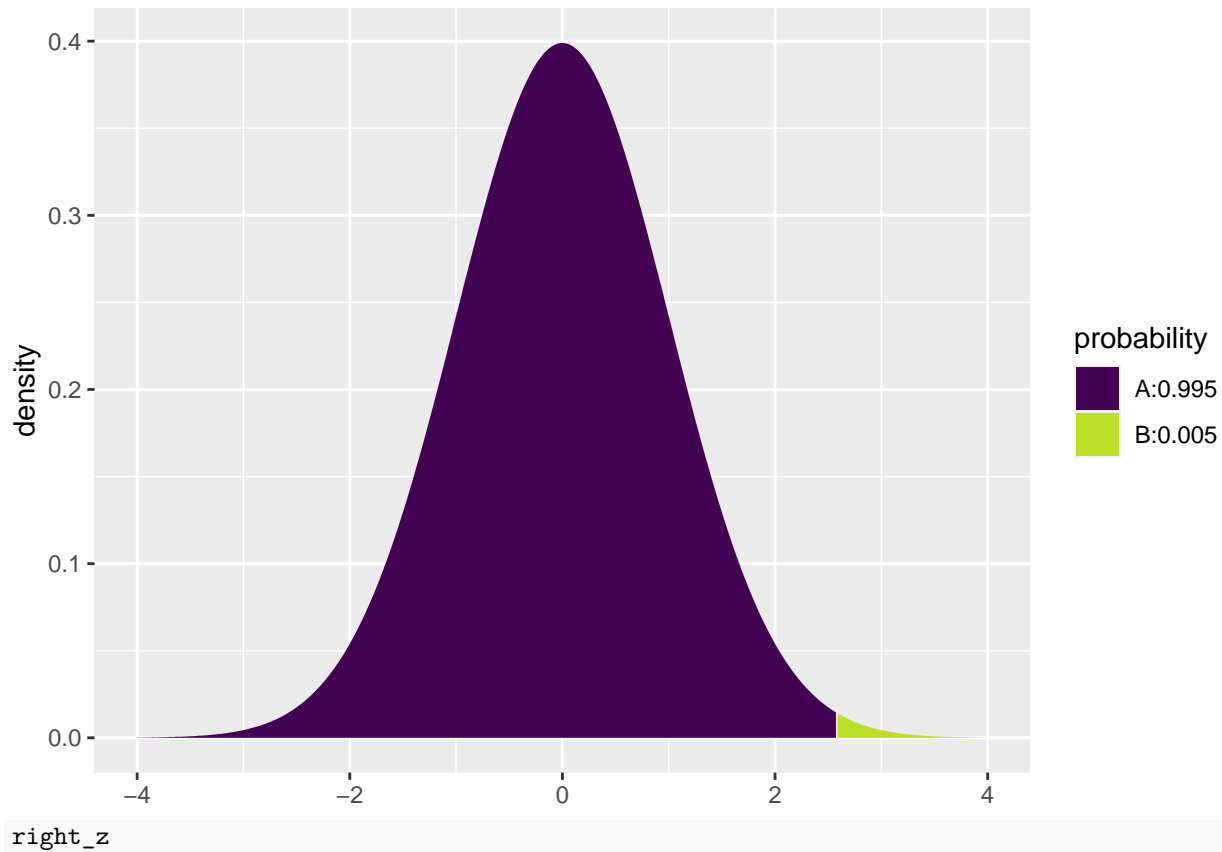


```
left_z
```

```
## [1] -2.6
```

- However, in all the formulas in the course we follow the textbook and consider  $z$ -values for a given right probability. E.g. with 0.5% probability to the right we get:

```
right_z <- qdist("norm", p = 1-0.005, mean = 0, sd = 1, xlim = c(-4, 4))
```



```
## [1] 2.6
```

- Thus, the probability of an observation between  $-2.576$  and  $2.576$  equals  $1 - 2 \cdot 0.005 = 99\%$ .

### 6.4.6 Example

The Stanford-Binet Intelligence Scale is calibrated to be approximately normal with mean 100 and standard deviation 16.

What is the 99-percentile of IQ scores?

- The corresponding  $z$ -score is  $Z = \frac{IQ-100}{16}$ , which means that  $IQ = 16Z + 100$ .
- The 99-percentile of  $z$ -scores has the value 2.326 (can be calculated using `qdist`).
- Then, the 99-percentile of IQ scores is:

$$IQ = 16 \cdot 2.326 + 100 = 137.2.$$

- So we expect that one out of hundred has an IQ exceeding 137.

## 7 Distribution of sample statistic

### 7.1 Estimates and their variability

We are given a sample  $y_1, y_2, \dots, y_n$ .

- The sample mean  $\bar{y}$  is the most common estimate of the population mean  $\mu$ .
- The sample standard deviation,  $s$ , is the most common estimate of the population standard deviation  $\sigma$ .

We notice that there is an uncertainty (from sample to sample) connected to these statistics and therefore we are interested in describing their **distribution**.

## 7.2 Distribution of sample mean

- We are given a sample  $y_1, y_2, \dots, y_n$  from a population with mean  $\mu$  and standard deviation  $\sigma$ .
- The sample mean

$$\bar{y} = \frac{1}{n}(y_1 + y_2 + \dots + y_n)$$

then has a distribution where

- the distribution has mean  $\mu$ ,
  - the distribution has standard deviation  $\frac{\sigma}{\sqrt{n}}$  (also called the **standard error**), and
  - when  $n$  grows, the distribution approaches a normal distribution. This result is called **the central limit theorem**.
- 

### 7.2.1 Central limit theorem

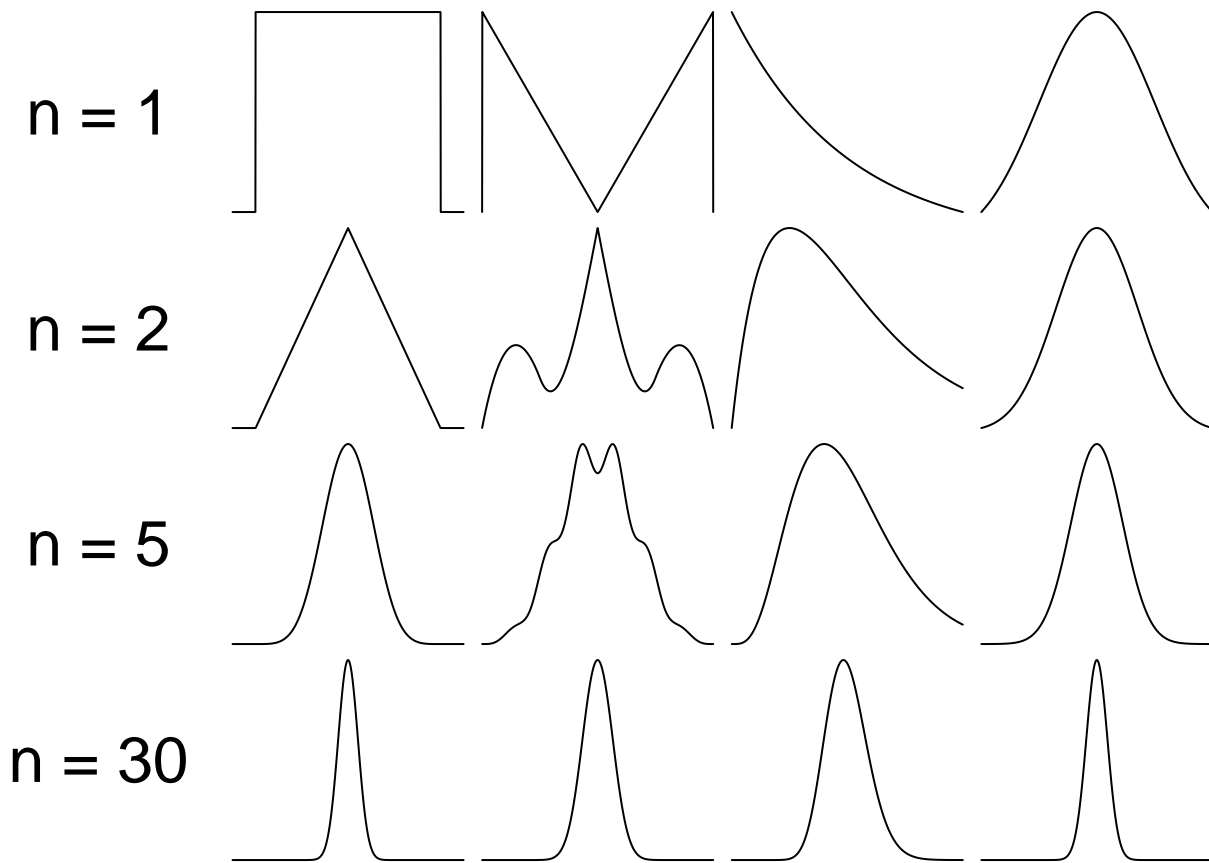
- The points above can be summarized as

$$\bar{y} \approx \text{norm}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

that is, based on the sample  $y_1, \dots, y_n$  of size  $n$  the sample mean  $\bar{y}$  is approximately normally distributed with mean  $\mu$  and standard error  $\frac{\sigma}{\sqrt{n}}$ .

- When our sample is sufficiently large (such that the above approximation is good) this allows us to make the following observations:
    - We are 95% certain that  $\bar{y}$  lies in the interval from  $\mu - 2\frac{\sigma}{\sqrt{n}}$  to  $\mu + 2\frac{\sigma}{\sqrt{n}}$ .
    - We are almost completely certain that  $\bar{y}$  lies in the interval from  $\mu - 3\frac{\sigma}{\sqrt{n}}$  to  $\mu + 3\frac{\sigma}{\sqrt{n}}$ .
  - This is not useful when  $\mu$  is unknown, but let us rephrase the first statement to:
    - We are 95% certain that  $\mu$  lies in the interval from  $\bar{y} - 2\frac{\sigma}{\sqrt{n}}$  to  $\bar{y} + 2\frac{\sigma}{\sqrt{n}}$ , i.e. we are directly talking about the uncertainty of determining  $\mu$ .
-

### 7.2.2 Illustration of CLT



- Four different population distributions ( $n=1$ ) of  $y$  and corresponding sampling distributions of  $\bar{y}$  for different sample sizes. As  $n$  increases the sampling distributions become narrower and more bell-shaped.

## 8 Estimation

### 8.1 Aim of statistics

- Statistics is all about “saying something” about a population.
- Typically, this is done by taking a random sample from the population.
- The sample is then analysed and a statement about the population can be made.
- The process of making conclusions about a population from analysing a sample is called **statistical inference**.

### 8.2 Random sampling schemes

Simple sampling(explained in Agresti section 2.2): Each experimental unit(fex persons) has the same probability of being selected(fex for an interview)

Other strategies for obtaining a random sample from the target population are explained in Agresti section 2.4:

- Systematic sampling
- Stratified sampling
- Cluster sampling
- Multistage sampling
- ...

### 8.3 Point and interval estimates

- We want to study hypotheses for population parameters, e.g. the mean  $\mu$  and the standard deviation  $\sigma$ .
  - If  $\mu$  is e.g. the mean waiting time in a queue, then it might be relevant to investigate whether it exceeds 2 minutes.
- Based on a sample we make a **point estimate** which is a guess of the parameter value.
  - For instance we have used  $\bar{y}$  as an estimate of  $\mu$  and  $s$  as an estimate of  $\sigma$ .
- We often want to supplement the point estimate with an **interval estimate** (also called a **confidence interval**). This is an interval around the point estimate, in which we are confident (to a certain degree) that the population parameter is located.
- The parameter estimate can then be used to investigate our hypothesis.

### 8.4 Point estimators: Bias

- If we want to estimate the population mean  $\mu$  we have several possibilities e.g.
  - the sample mean  $\bar{y}$
  - the average  $y_T$  of the sample upper and lower quartiles
- Advantage of  $y_T$ : Very large/small observations have little effect, i.e. it has practically no effect if there are a few errors in the data set.
- Disadvantage of  $y_T$ : If the distribution of the population is skewed, i.e. asymmetrical, then  $y_T$  is **biased**, meaning that in the long run this estimator systematically over or under estimates the value of  $\mu$ .
- Generally we prefer that an estimator is **unbiased**, i.e. its distribution is centered around the true parameter value.
- Recall that for a sample from a population with mean  $\mu$ , the sample mean  $\bar{y}$  also has mean  $\mu$ . That is,  $\bar{y}$  is an unbiased estimate of the population mean  $\mu$ .

### 8.5 Point estimators: Consistency

- From previous lectures we know that the standard error of  $\bar{y}$  is  $\frac{\sigma}{\sqrt{n}}$ ,
  - i.e. the standard error decrease when the sample size increase.
- In general an estimator with this property is called **consistent**.
- $y_T$  is also a consistent estimator, but has a variance that is greater than  $\bar{y}$ .

### 8.6 Point estimators: Efficiency

- Since the variance of  $y_T$  is greater than the variance of  $\bar{y}$ ,  $\bar{y}$  is preferred.
- In general we prefer the estimator with the smallest possible variance.
  - This estimator is said to be **efficient**.
- $\bar{y}$  is an efficient estimator.

### 8.7 Notation

- The symbol  $\hat{\cdot}$  above a parameter is often used to denote a (point) estimate of the parameter. We have looked at an
  - estimate of the population mean:  $\hat{\mu} = \bar{y}$
  - estimate of the population standard deviation:  $\hat{\sigma} = s$
- When we observe a 0/1 variable, which e.g. is used to denote yes/no or male/female, then we will use the notation

$$\pi = P(Y = 1)$$

for the proportion of the population with the property  $Y = 1$ .

- The estimate  $\hat{\pi} = (y_1 + y_2 + \dots + y_n)/n$  is the relative frequency of the property  $Y = 1$  in the sample.

### 8.8 Confidence Interval

- The general definition of a confidence interval for a population parameter is as follows:

- A **confidence interval** for a parameter is constructed as an interval, where we expect the parameter to be.
- The probability that this construction yields an interval which includes the parameter is called the **confidence level** and it is typically chosen to be 95%.
- (1-confidence level) is called the **error probability** (in this case  $1 - 0.95 = 0.05$ , i.e. 5%).
- In practice the interval is often constructed as a symmetric interval around a point estimate:
  - **point estimate ± margin of error**
  - Rule of thumb: With a margin of error of 2 times the standard error you get a confidence interval, where the confidence level is approximately 95%.
  - I.e: **point estimate ± 2 x standard error** has confidence level of approximately 95%.

## 8.9 Confidence interval for proportion

- Consider a population with a distribution where the probability of having a given characteristic is  $\pi$  and the probability of not having it is  $1 - \pi$ .
- When *no/yes* to the characteristic is denoted 0/1, i.e.  $y$  is 0 or 1, the distribution of  $y$  have a standard deviation of:

$$\sigma = \sqrt{\pi(1 - \pi)}.$$

That is, the standard deviation is not a “free” parameter for a 0/1 variable as it is directly linked to the probability  $\pi$ .

- With a sample size of  $n$  the standard error of  $\hat{\pi}$  will be (since  $\hat{\pi} = \frac{\sum_{i=1}^n y_i}{n}$ ):

$$\sigma_{\hat{\pi}} = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{\pi(1 - \pi)}{n}}.$$

- We do not know  $\pi$  but we insert the estimate and get the **estimated standard error** of  $\hat{\pi}$ :

$$se = \sqrt{\frac{\hat{\pi}(1 - \hat{\pi})}{n}}.$$

- The rule of thumb gives that the interval

$$\hat{\pi} \pm 2\sqrt{\frac{\hat{\pi}(1 - \hat{\pi})}{n}}$$

has confidence level of approximately 95%. I.e., before the data is known the random interval given by the formula above has approximately 95% probability of covering the true value  $\pi$ .

### 8.9.1 Example: Point and interval estimate for proportion

- Now we will have a look at a data set concerning votes in Chile. Information about the data can be found here.

```
Chile <- read.delim("https://asta.math.aau.dk/datasets?file=Chile.txt")
```

- We focus on the variable **sex**, i.e. the gender distribution in the sample.

```
library(mosaic)
tally(~ sex, data = Chile)
```

```
## sex
##   F   M
## 1379 1321
```



```
tally(~ sex, data = Chile, format = "prop")
```

```
## sex
##   F   M
## 0.51 0.49
```

- Unknown population proportion of females (F),  $\pi$ .
  - Estimate of  $\pi$ :  $\hat{\pi} = \frac{1379}{1379+1321} = 0.5107$
  - Rule of thumb :  $\hat{\pi} \pm 2 \times se = 0.5107 \pm 2\sqrt{\frac{0.5107(1-0.5107)}{1379+1321}} = (0.49, 0.53)$  is an approximate 95% confidence interval for  $\pi$ .
- 

### 8.9.2 Example: Confidence intervals for proportion in R

- **R** automatically calculates the confidence interval for the proportion of females when we do a so-called hypothesis test (we will get back to that later):

```
prop.test(~ sex, data = Chile, correct = FALSE)
```

```
##
## 1-sample proportions test without continuity correction
##
## data:  Chile$sex [with success = F]
## X-squared = 1, df = 1, p-value = 0.3
## alternative hypothesis: true p is not equal to 0.5
## 95 percent confidence interval:
##  0.49 0.53
## sample estimates:
##      p
## 0.51
```

- The argument `correct = FALSE` is needed to make **R** use the “normal” formulas as on the slides and in the book. When `correct = TRUE` (the default) a mathematical correction which you have not learned about is applied and slightly different results are obtained.
- 

## 8.10 General confidence intervals for proportion

- Based on the central limit theorem we have:

$$\hat{\pi} \approx N\left(\pi, \sqrt{\frac{\pi(1-\pi)}{n}}\right)$$

if  $n\hat{\pi}$  and  $n(1-\hat{\pi})$  both are at least 15.

- To construct a confidence interval with (approximate) confidence level  $1 - \alpha$ :
    - 1) Find the so-called **critical value**  $z_{crit}$  for which the upper tail probability in the standard normal distribution is  $\alpha/2$ .
    - 2) Calculate  $se = \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$
    - 3) Then  $\hat{\pi} \pm z_{crit} \times se$  is a confidence interval with confidence level  $1 - \alpha$ .
-

### 8.10.1 Example: Chile data

Compute for the **Chile** data set the 99% and 95%-confidence intervals for the probability that a person is female:

- For a 99%-confidence level we have  $\alpha = 1\%$  and
  - 1)  $z_{crit} = \text{qdist}(\text{"norm"}, 1 - 0.01/2) = 2.576$ .
  - 2) We know that  $\hat{\pi} = 0.5107$  and  $n = 2700$ , so  $se = \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} = 0.0096$ .
  - 3) Thereby, a 99%-confidence interval is:  $\hat{\pi} \pm z_{crit} \times se = (0.4859, 0.5355)$ .
- For a 95%-confidence level we have  $\alpha = 5\%$  and
  - 1)  $z_{crit} = \text{qdist}(\text{"norm"}, 1 - 0.05/2) = 1.96$ .
  - 2) Again,  $\hat{\pi} = 0.5107$  and  $n = 2700$  and so  $se = 0.0096$ .
  - 3) Thereby, we find as 95%-confidence interval  $\hat{\pi} \pm z_{crit} \times se = (0.4918, 0.5295)$  (as the result of `prop.test`).

## 8.11 Confidence Interval for mean - normally distributed sample

- When it is reasonable to assume that the population distribution is normal we have the **exact** result

$$\bar{y} \sim \text{norm}(\mu, \frac{\sigma}{\sqrt{n}}),$$

i.e.  $\bar{y} \pm z_{crit} \times \frac{\sigma}{\sqrt{n}}$  is not only an approximate but rather an exact confidence interval for the population mean,  $\mu$ .

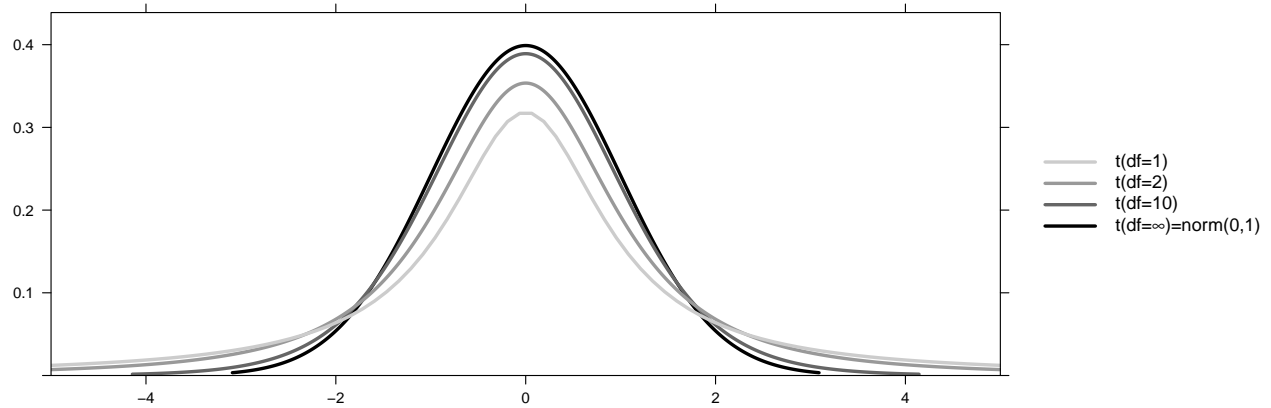
- In practice we **do not know**  $\sigma$  and instead we are forced to apply the sample standard deviation  $s$  to find the **estimated standard error**  $se = \frac{s}{\sqrt{n}}$ .
- This extra uncertainty, however, implies that an exact confidence interval for the population mean  $\mu$  cannot be constructed using the  $z$ -score.
- Luckily, an exact interval can still be constructed by using the so-called  **$t$ -score**, which apart from the confidence level depends on the **degrees of freedom**, which are  $df = n - 1$ . That is the confidence interval now takes the form

$$\bar{y} \pm t_{crit} \times se.$$

## 8.12 $t$ -distribution and $t$ -score

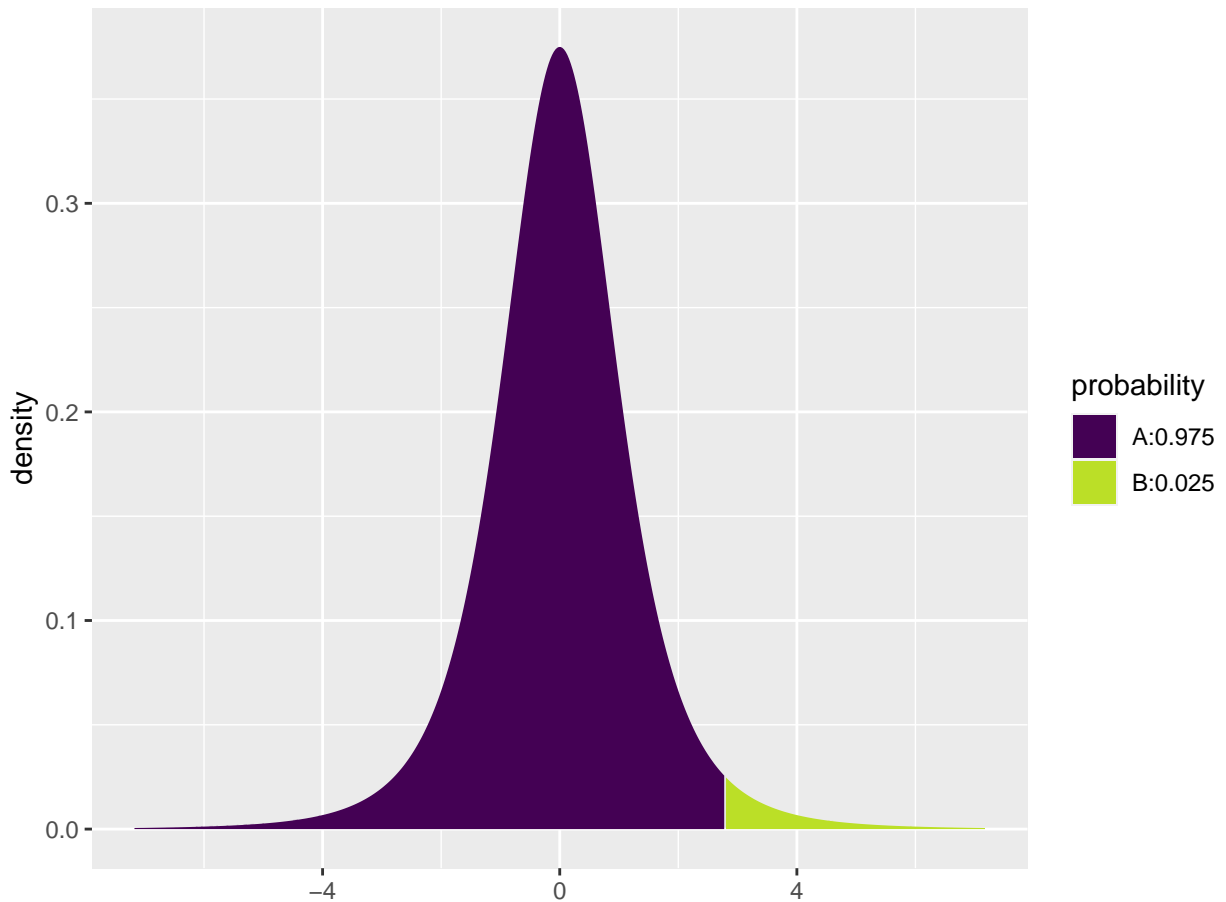
- Calculation of  $t$ -score is based on the  **$t$ -distribution**, which is very similar to the standard normal  $z$ -distribution:
  - it is symmetric around zero and bell shaped, but
  - it has “heavier” tails and thereby
  - a slightly larger standard deviation than the standard normal distribution.
  - Further, the  $t$ -distribution’s standard deviation decays as a function of its **degrees of freedom**, which we denote  $df$ .
  - and when  $df$  grows the  $t$ -distribution approaches the standard normal distribution.

The expression of the density function is of slightly complicated form and will not be stated here, instead the  $t$ -distribution is plotted below for  $df = 1, 2, 10$  and  $\infty$ .



### 8.13 Calculation of $t$ -score in R

```
qdist("t", p = 1 - 0.025, df = 4)
```



```
## [1] 2.8
```

- We seek the quantile (i.e. value on the x-axis) such that we have a given **right tail** probability. This is the critical  $t$ -score associated with our desired level of confidence.
- To get e.g. the  $t$ -score corresponding to a right tail probability of 2.5 % we have to look up the 97.5 % quantile using `qdist` with  $p = 1 - 0.025$  since `qdist` looks at the area to the **left hand side**.

- The degrees of freedom are determined by the sample size; in this example we just used  $df = 4$  for illustration.
- As the  $t$ -score giving a right probability of 2.5 % is 2.776 and the  $t$ -distribution is symmetric around 0, we have that an observation falls between -2.776 and 2.776 with probability  $1 - 2 \cdot 0.025 = 95$  % for a  $t$ -distribution with 4 degrees of freedom.

### 8.14 Example: Confidence interval for mean

- We return to the dataset `Ericksen` and want to construct a 95% confidence interval for the population mean  $\mu$  of the variable `crime`.

```
Ericksen <- read.delim("https://asta.math.aau.dk/datasets?file=Ericksen.txt")
stats <- favstats( ~ crime, data = Ericksen)
stats
```

```
## min Q1 median Q3 max mean sd n missing
## 25 48 55 73 143 63 25 66 0
```

```
qdist("t", 1 - 0.025, df = 66 - 1, plot = FALSE)
```

```
## [1] 2
```

- I.e. we have
  - $\bar{y} = 63.06$
  - $s = 24.89$
  - $n = 66$
  - $df = n - 1 = 65$
  - $t_{crit} = 2$ .
- The confidence interval is  $\bar{y} \pm t_{crit} \frac{s}{\sqrt{n}} = (56.94, 69.18)$
- All these calculations can be done automatically by **R**:

```
t.test( ~ crime, data = Ericksen, conf.level = 0.95)
```

```
##
## One Sample t-test
##
## data: crime
## t = 21, df = 65, p-value <2e-16
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
## 57 69
## sample estimates:
## mean of x
## 63
```

### 8.15 Example: Plotting several confidence intervals in R

- We shall look at a built-in **R** dataset `chickwts`.
- `?chickwts` yields a page with the following information

An experiment was conducted to measure and compare the effectiveness of various feed supplements on the growth rate of chickens. Newly hatched chicks were randomly allocated into six groups, and each group was given a different feed supplement. Their weights in grams after six weeks are given along with feed types.

- `chickwts` is a data frame with 71 observations on 2 variables:
  - `weight`: a numeric variable giving the chick weight.

– feed: a factor giving the feed type.

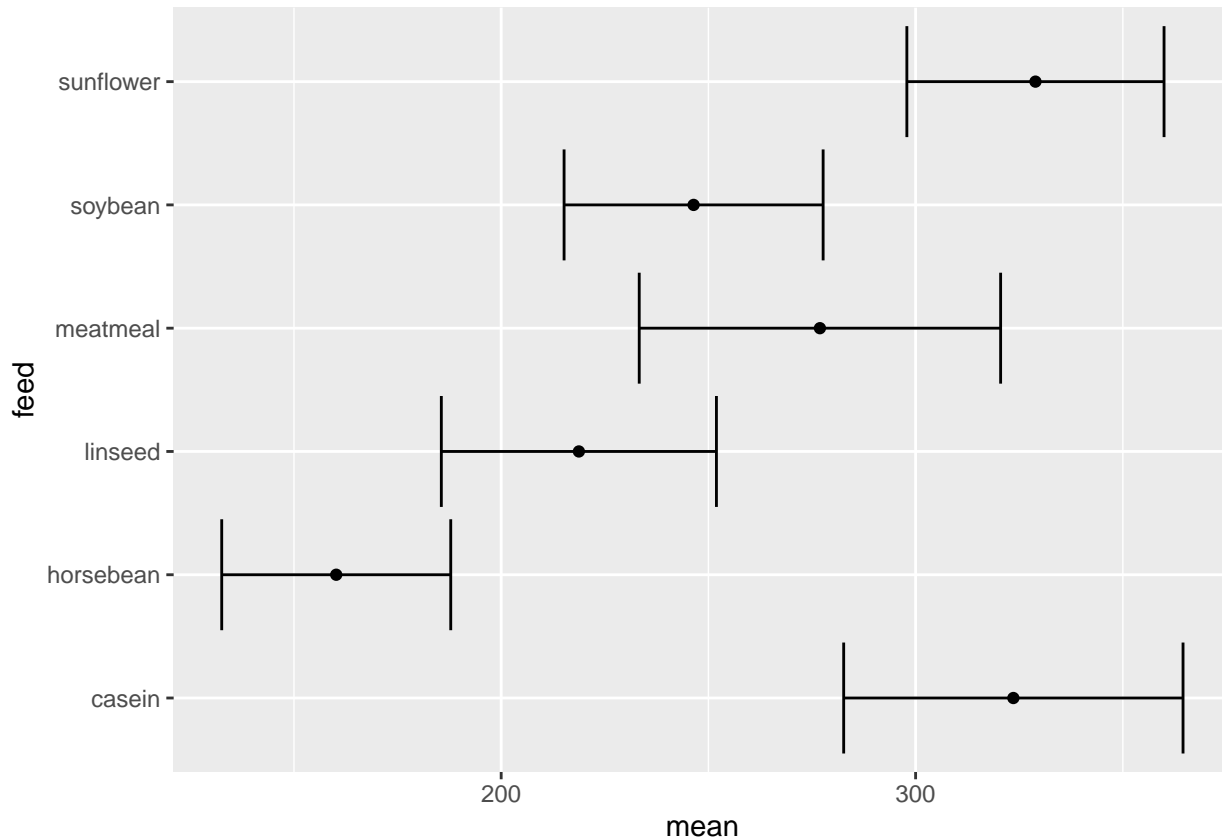
- Calculate a confidence interval for the mean weight for each feed separately; the confidence interval is from lower to upper given by  $\text{mean} \pm \text{tscore} * \text{se}$ :

```
cwei <- favstats( weight ~ feed, data = chickwts)
se <- cwei$sd / sqrt(cwei$n) # Standard errors
tscore <- qdist("t", p = .975, df = cwei$n - 1, plot = FALSE) # t-scores for 2.5% right tail probability
cwei$lower <- cwei$mean - tscore * se
cwei$upper <- cwei$mean + tscore * se
cwei[, c("feed", "mean", "lower", "upper")]
```

```
##      feed mean lower upper
## 1  casein  324   283   365
## 2 horsebean 160   133   188
## 3  linseed  219   186   252
## 4  meatmeal 277   233   321
## 5  soybean  246   215   278
## 6 sunflower 329   298   360
```

- We can plot the confidence intervals as horizontal line segments using `gf_errorbarh`:

```
gf_errorbarh(feed ~ lower + upper, data = cwei) %>%
  gf_point(feed ~ mean)
```



## 9 Determining sample size

### 9.1 Sample size for proportion

- The confidence interval is of the form point estimate  $\pm$  estimated margin of error.
- When we estimate a proportion the margin of error is

$$M = z_{crit} \sqrt{\frac{\pi(1-\pi)}{n}},$$

where the critical  $z$ -score,  $z_{crit}$ , is determined by the specified confidence level.

- Imagine that we want to plan an experiment, where we **want to achieve a certain margin of error**  $M$  (and thus a specific width of the associated confidence interval).
- If we solve the equation above we see:
  - If we choose sample size  $n = \pi(1-\pi)\left(\frac{z_{crit}}{M}\right)^2$ , then we obtain an estimate of  $\pi$  with margin of error  $M$ .
- If we do not have a good guess for the value of  $\pi$  we can use the worst case value  $\pi = 50\%$ . The corresponding sample size  $n = \left(\frac{z_{crit}}{2M}\right)^2$  ensures that we obtain an estimate with a margin of error, which is at the *most*  $M$ .

---

#### 9.1.1 Example

- Let us choose  $z_{crit} = 1.96$ , i.e the confidence level is 95%.
- How many voters should we ask to get a margin of error, which equals 1%?
- Worst case is  $\pi = 0.5$ , yielding:

$$n = \pi(1-\pi) \left(\frac{z_{crit}}{M}\right)^2 = \frac{1}{4} \left(\frac{1.96}{0.01}\right)^2 = 9604.$$

- If we are interested in the proportion of voters that vote for "socialdemokratiet", a good guess is  $\pi = 0.23$ , yielding

$$n = \pi(1-\pi) \left(\frac{z_{crit}}{M}\right)^2 = 0.23(1-0.23) \left(\frac{1.96}{0.01}\right)^2 = 6804.$$

- If we instead are interested in "liberal alliance", a good guess is  $\pi = 0.05$ , yielding

$$n = \pi(1-\pi) \left(\frac{z_{crit}}{M}\right)^2 = 0.05(1-0.05) \left(\frac{1.96}{0.01}\right)^2 = 1825.$$

### 9.2 Sample size for mean

- The confidence interval is of the form point estimate  $\pm$  estimated margin of error.
- When we estimate a mean the margin of error is

$$M = z_{crit} \frac{\sigma}{\sqrt{n}},$$

where the critical  $z$ -score,  $z_{crit}$ , is determined by the specified confidence level.

- Imagine that we want to plan an experiment, where we **want to achieve a certain margin of error**  $M$ .
- If we solve the equation above we see:
  - If we choose sample size  $n = \left(\frac{z_{crit}\sigma}{M}\right)^2$ , then we obtain an estimate with margin of error  $M$ .
- Problem: We usually do not know  $\sigma$ . Possible solutions:
  - Based on similar studies conducted previously, we make a qualified guess at  $\sigma$ .
  - Based on a pilot study a value of  $\sigma$  is estimated.