ASTA

The ASTA team

Contents

1 Statistical inference: Hypothesis and test

1.1 Concept of hypothesis

- A **hypothesis** is a statement about a given population. Usually it is stated as a population parameter having a given value or being in a certain interval.
- Examples:
  - Quality control of products: The hypothesis is that the products e.g. have a certain weight, a given power consumption or a minimal durability.
  - Scientific hypothesis: There is no dependence between a company's age and level of return.

1.2 Significance test

- A significance test is used to investigate, whether data is contradicting the hypothesis or not.
- If the hypothesis says that a parameter has a certain value, then the test should tell whether the sample estimate is "far" away from this value.
- For example:
  - Waiting times in a queue. We sample n customers and count how many that have been waiting more than 5 minutes. The company policy is that at most 10% of the customers should wait more than 5 minutes. In a sample of size $n = 32$ we observe 4 with waiting time above 5 minutes, i.e. the estimated proportion is $\hat{\pi} = \frac{4}{32} = 12.5\%$. Is this "much more" than (i.e. significantly different from) 10%?
  - The blood alcohol level of a student is measured 4 times with the values $0.504, 0.500, 0.512, 0.524$, i.e. the estimated mean value is $\bar{y} = 0.51$. Is this "much different" than a limit of 0.5?

1.3 Null and alternative hypothesis

- **The null hypothesis** - denoted H_0 - usually specifies that a population parameter has some given value. E.g. if μ is the mean blood alcohol level we can state the null hypothesis
  - $H_0 : \mu = 0.5$.
- The **alternative hypothesis** - denoted H_a - usually specifies that the population parameter is contained in a given set of values different than the null hypothesis. E.g. if μ again is the population mean of a blood alcohol level measurement, then
  - the null hypothesis is $H_0 : \mu = 0.5$
  - the alternative hypothesis is $H_a : \mu \neq 0.5$.

1.4 Test statistic

- We consider a population parameter μ and write the null hypothesis

$$H_0 : \mu = \mu_0,$$

  where μ_0 is a known number, e.g. $\mu_0 = 0.5$.
- Based on a sample we have an estimate $\hat{\mu}$.
- A **test statistic** T will typically depend on $\hat{\mu}$ and μ_0 (we may write this as $T(\hat{\mu}, \mu_0)$) and measures "how far from μ_0 is $\hat{\mu}$?"
- Often we use $T(\hat{\mu}, \mu_0)$ = "the number of standard deviations from $\hat{\mu}$ to μ_0".
- For example it would be very unlikely to be more than 3 standard deviations from μ_0, i.e. in that case μ_0 is probably not the correct value of the population parameter.

1.5 P-value

- We consider
  - H_0: a null hypothesis.
  - H_a: an alternative hypothesis.
  - T: a test statistic, where the value calculated based on the current sample is denoted t_{obs}.
- To investigate the plausibility of H_0, we measure the evidence against H_0 by the so-called p-value:
  - The p-value is the probability of observing a more extreme value of T (if we were to repeat the experiment) than t_{obs} *under the assumption that* H_0 *is true.*

  - "Extremity" is measured relative to the alternative hypothesis; a value is considered extreme if it is "far from" H_0 and "closer to" H_a.
  - If the p-value is small then there is a small probability of observing t_{obs} if H_0 is true, and thus H_0 is not very probable for our sample and we put more support in H_a, so:

The smaller the p-value, the less we trust H_0.

- What is a small p-value? If it is below 5% we say it is **significant** at the 5% level.

1.6 Significance level

- We consider
  - H_0: a null hypothesis.
  - H_a: an alternative hypothesis.
  - T: a test statistic, where the value calculated based on the current sample is denoted t_{obs} and the corresponding p-value is p_{obs}.
- Small values of p_{obs} are critical for H_0.
- In practice it can be necessary to decide whether or not we are going to reject H_0.
- The decision can be made if we previously have decided on a so-called α**-level**, where
  - α is a given percentage
  - we reject H_0, if p_{obs} is less than or equal to α
  - α is called the **significance level** of the test
  - typical choices of α are 5% or 1%.

1.7 Significance test for mean

1.7.1 Two-sided t-test for mean:

- We assume that data is a sample from $\texttt{norm}(\mu, \sigma)$.
- The estimates of the population parameters are $\hat{\mu} = \bar{y}$ and $\hat{\sigma} = s$ based on n observations.
- Null hypothesis: $H_0 : \ \mu = \mu_0$, where μ_0 is a known value.
- **Two-sided alternative hypothesis**: $H_a : \ \mu \neq \mu_0$.
- Observed test statistic: $t_{obs} = \frac{\bar{y} - \mu_0}{se}$, where $se = \frac{s}{\sqrt{n}}$.
- I.e. t_{obs} measures, how many standard deviations (with $\pm$ sign) the empirical mean lies away from μ_0.
- If H_0 is true, then t_{obs} is an observation from the t-distribution with $df = n - 1$.
- P-value: Values bigger than $|t_{obs}|$ or less than $-|t_{obs}|$ puts more support in H_a than H_0.
- The p-value = 2 x "upper tail probability of $|t_{obs}|$". The probability is calculated in the t-distribution with df degrees of freedom.

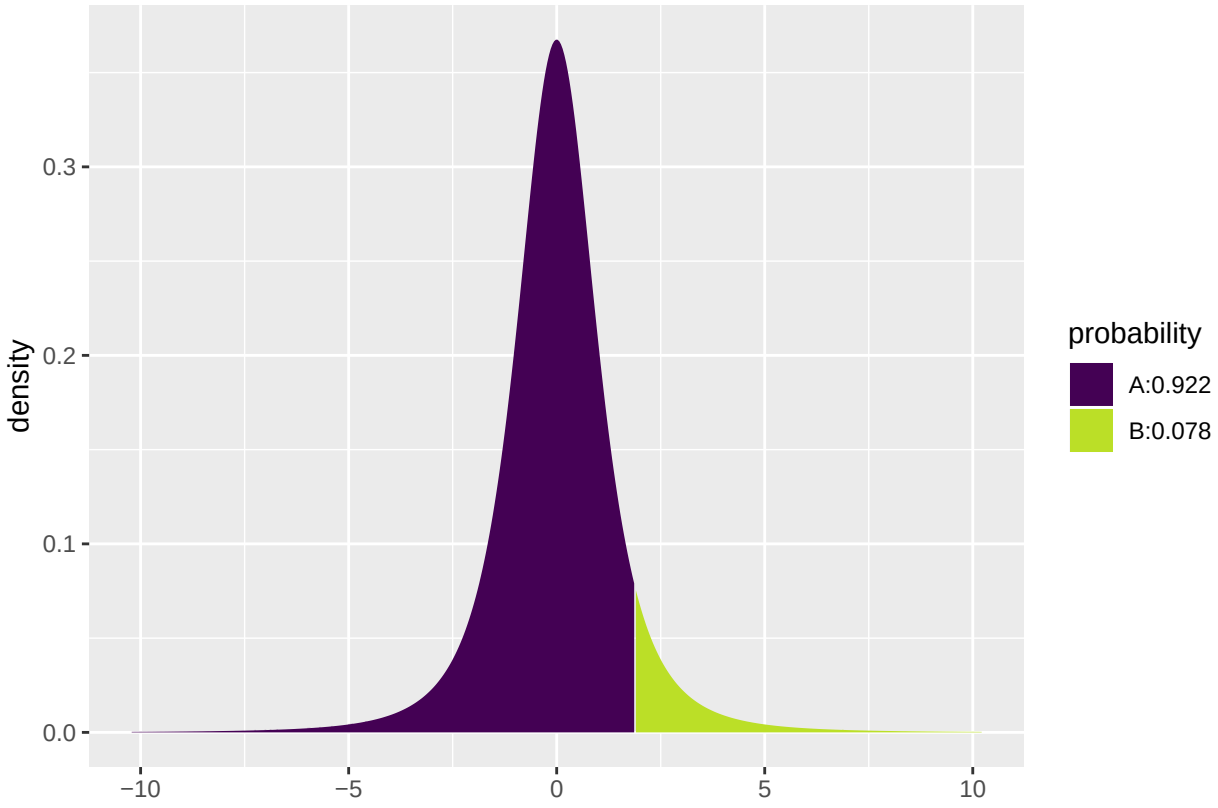
1.7.2 Example: Two-sided t-test

- Blood alcohol level measurements: $0.504, 0.500, 0.512, 0.524$.
- These are assumed to be a sample from a normal distribution.
- We calculate
  - $\bar{y} = 0.51$ and $s = 0.0106$
  - $se = \frac{s}{\sqrt{n}} = \frac{0.0106}{\sqrt{4}} = 0.0053$.
  - $H_0 : \mu = 0.5$, i.e. $\mu_0 = 0.5$.

- $t_{obs} = \frac{\bar{y}-\mu_0}{se} = \frac{0.51-0.5}{0.0053} = 1.89$.

- So we are almost 2 standard deviations from 0.5. Is this extreme in a t-distribution with 3 degrees of freedom?

```
library(mosaic)
1 - pdist("t", q = 1.89, df = 3)
```




```
## [1] 0.07757725
```

- The p-value is $2 \cdot 0.078$, i.e. more than 15%. On the basis of this we do not reject H_0.

1.8 One-sided t-test for mean

The book also discusses one-sided t-tests for the mean, but we will not use those in the course.

1.9 Agresti: Overview of t-test

TABLE 6.3: The Five Parts of Significance Tests for Population Means

1. **Assumptions**
   Quantitative variable
   Randomization
   Normal population (robust, especially for two-sided H_a, large n)
2. **Hypotheses**
   H_0: $\mu = \mu_0$
   H_a: $\mu \neq \mu_0$ (or H_a: $\mu > \mu_0$ or H_a: $\mu < \mu_0$)
3. **Test statistic**
   $t = \dfrac{\bar{y} - \mu_0}{se}$ where $se = \dfrac{s}{\sqrt{n}}$
4. ***P*-value**
   In t curve, use
   P = Two-tail probability for H_a: $\mu \neq \mu_0$
   P = Probability to right of observed t-value for H_a: $\mu > \mu_0$
   P = Probability to left of observed t-value for H_a: $\mu < \mu_0$
5. **Conclusion**
   Report P-value. Smaller P provides stronger evidence against H_0 and supporting H_a. Can reject H_0 if $P \leq \alpha$-level.

1.10 Significance test for proportion

- Consider a sample of size n, where we observe whether a given property is present or not.
- The relative frequency of the property in the population is π, which is estimated by $\hat{\pi}$.
- Null hypothesis: $H_0 : \pi = \pi_0$, where π_0 is a known number.
- **Two-sided alternative** hypothesis: $H_a : \pi \neq \pi_0$.
- *If H_0 is true* the standard error for $\hat{\pi}$ is given by $se_0 = \sqrt{\frac{\pi_0(1-\pi_0)}{n}}$.
- Observed test statistic: $z_{obs} = \frac{\hat{\pi}-\pi_0}{se_0}$
- I.e. z_{obs} measures, how many standard deviations (with $\pm$ sign) there is from $\hat{\pi}$ to π_0.

1.10.1 Approximate test

- If both $n\hat{\pi}$ and $n(1-\hat{\pi})$ are larger than 15 we know from previously that $\hat{\pi}$ follows a normal distribution (approximately), i.e.
  - If H_0 is true, then z_{obs} is an observation from the standard normal distribution.
- P-value for **two-sided** test: Values greater than $|z_{obs}|$ or less than $-|z_{obs}|$ point more towards H_a than H_0.
- The p-value=2 x "upper tail probability for $|z_{obs}|$". The probability is calculated in the standard normal distribution.

1.10.2 Example: Approximate test

- We consider a study from Florida Poll 2006:
  - In connection with problems financing public service a random sample of 1200 individuals were asked whether they preferred less service or tax increases.
  - 52% preferred tax increases. Is this enough to say that the proportion is significantly different from fifty-fifty?
- Sample with $n = 1200$ observations and estimated proportion $\hat{\pi} = 0.52$.
- Null hypothesis $H_0 : \pi = 0.5$.
- Alternative hypothesis $H_a : \pi \neq 0.5$.
- Standard error $se_0 = \sqrt{\frac{\pi_0(1-\pi_0)}{n}} = \sqrt{\frac{0.5 \times 0.5}{1200}} = 0.0144$
- Observed test statistic $z_{obs} = \frac{\hat{\pi}-\pi_0}{se_0} = \frac{0.52-0.5}{0.0144} = 1.39$
- "upper tail probability for 1.39" in the standard normal distribution is 0.0823, i.e. we have a p-value of $2 \cdot 0.0823 \approx 16\%$.
- Conclusion: There is not sufficient evidence to reject H_0, i.e. we do not reject that the preference in the population is fifty-fifty.
- Note, the above calculations can also be performed automatically in **R** by (a little different results due to rounding errors in the manual calculation):

```
count <- 1200 * 0.52 # number of individuals preferring tax increase
prop.test(x = count, n = 1200, correct = F)
```

```
##
##  1-sample proportions test without continuity correction
##
## data:  count out of 1200
## X-squared = 1.92, df = 1, p-value = 0.1659
## alternative hypothesis: true p is not equal to 0.5
## 95 percent confidence interval:
##  0.4917142 0.5481581
## sample estimates:
##    p
## 0.52
```

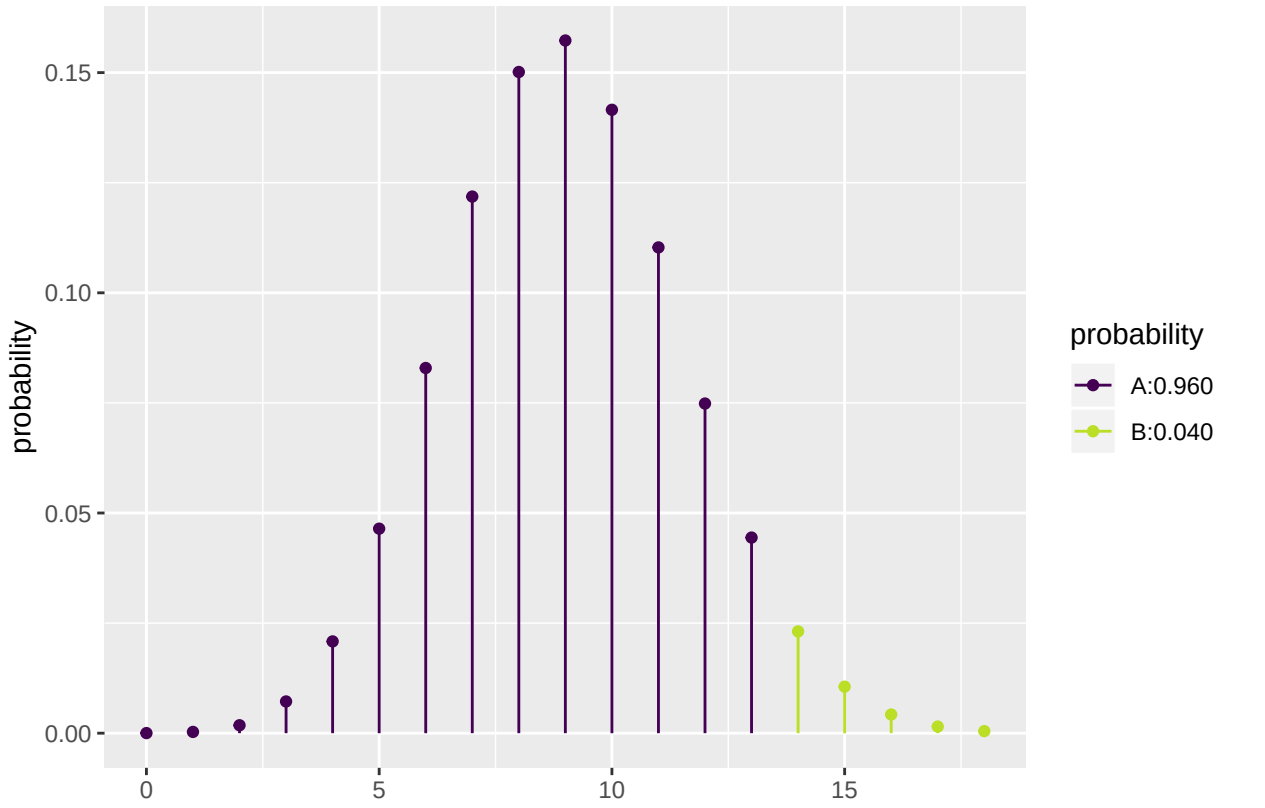
1.10.3 Binomial (exact) test

- Consider again a sample of size n, where we observe whether a given property is present or not.
- The relative frequency of the property in the population is π, which is estimated by $\hat{\pi}$.
- Let $y_+ = n\hat{\pi}$ be the frequency (total count) of the property in the sample.
- It can be shown that y_+ follows the **binomial distribution** with size parameter n and success probability π. We use $Bin(n, \pi)$ to denote this distribution.
- Null hypothesis: $H_0 : \pi = \pi_0$, where π_0 is a known number.
- Alternative hypothesis: $H_a : \pi \neq \pi_0$, where π_0 is a known number.
- P-value for **two-sided** binomial test:
  - If $y_+ \geq n\pi_0$: 2 x "upper tail probability for y_+" in the $Bin(n, \pi_0)$ distribution.
  - If $y_+ < n\pi_0$: 2 x "lower tail probability for y_+" in the $Bin(n, \pi_0)$ distribution.

1.10.4 Example: Binomial test

- Experiment with $n = 30$, where we have $y_+ = 14$ successes.
- We want to test $H_0 : \pi = 0.3$ vs. $H_a : \pi \neq 0.3$.
- Since $y_+ > n\pi_0 = 9$ we use the upper tail probability corresponding to the sum of the height of the red lines to the right of 14 in the graph below. (Notice, the graph continues on the right hand side to $n = 30$, but it has been cut off for illustrative purposes.)
- The upper tail probability from 14 and up (i.e. greater than 13) is:

```
lower_tail <- pdist("binom", q = 13, size = 30, prob = 0.3)
```




```
1 - lower_tail
```

```
## [1] 0.04005255
```

- The two-sided p-value is then 2 x 0.04 = 0.08.

1.10.5 Binomial test in R

- We return to the Chile data, where we again look at the variable `sex`.
- Let us test whether the proportion of females is different from 50 %, i.e., we look at $H_0 : \pi = 0.5$ and $H_a : \pi \neq 0.5$, where π is the unknown population proportion of females.

```
Chile <- read.delim("https://asta.math.aau.dk/datasets?file=Chile.txt")
binom.test( ~ sex, data = Chile, p = 0.5, conf.level = 0.95)
```

```
##
##
##
## data:  Chile$sex  [with success = F]
## number of successes = 1379, number of trials = 2700, p-value =
## 0.2727
## alternative hypothesis: true probability of success is not equal to 0.5
## 95 percent confidence interval:
##  0.4916971 0.5297610
## sample estimates:
## probability of success
##              0.5107407
```

- The p-value for the binomial exact test is 27%, so there is no significant difference between the proportion of males and females.
- The approximate test has a p-value of 26%, which can be calculated by the command

```
prop.test( ~ sex, data = Chile, p = 0.5, conf.level = 0.95, correct = FALSE)
```

(note the additional argument `correct = FALSE`).

1.11 Agresti: Overview of tests for mean and proportion

TABLE 6.7: Summary of Significance Tests for Means and Proportions

Parameter	Mean	Proportion
1. Assumptions	Random sample, quantitative variable normal population	Random sample, categorical variable null expected counts at least 10
2. Hypotheses	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$ $H_a: \mu > \mu_0$ $H_a: \mu < \mu_0$	$H_0: \pi = \pi_0$ $H_a: \pi \neq \pi_0$ $H_a: \pi > \pi_0$ $H_a: \pi < \pi_0$
3. Test statistic	$t = \frac{\bar{y} - \mu_0}{se}$ with $se = \frac{s}{\sqrt{n}}, df = n - 1$	$z = \frac{\hat{\pi} - \pi_0}{se_0}$ with $se_0 = \sqrt{\pi_0(1 - \pi_0)/n}$
4. *P*-value	Two-tail probability in sampling distribution for two-sided test ($H_0: \mu \neq \mu_0$ or $H_a: \pi \neq \pi_0$); One-tail probability for one-sided test	
5. Conclusion	Reject H_0 if *P*-value $\leq \alpha$-level such as 0.05	

1.12 Response variable and explanatory variable

- We conduct an experiment, where we at random choose 50 IT-companies and 50 service companies and measure their profit ratio. Is there association between company type (IT/service) and profit ratio?
- In other words we compare samples from 2 different populations. For each company we register:
  - The binary variable `company type`, which is called **the explanatory variable** and divides data in 2 groups.
  - The quantitative variable `profit ratio`, which is called **the response variable**.

1.13 Dependent/independent samples

- In the example with profit ratio of 50 IT-companies and 50 service companies we have **independent samples**, since the same company cannot be in both groups.
- Now, think of another type of experiment, where we at random choose 50 IT-companies and measure their profit ratio in both 2009 and 2010. Then we may be interested in whether there is association between year and profit ratio?
- In this example we have **dependent samples**, since the same company is in both groups.
- Dependent samples may also be referred to as paired samples.

1.14 Comparison of two means (Independent samples)

- We consider the situation, where we have two quantitative samples:
  - Population 1 has mean μ_1, which is estimated by $\hat{\mu}_1 = \bar{y}_1$ based on a sample of size n_1.
  - Population 2 has mean μ_2, which is estimated by $\hat{\mu}_2 = \bar{y}_2$ based on a sample of size n_2.
  - We are interested in the difference $\mu_2 - \mu_1$, which is estimated by $d = \bar{y}_2 - \bar{y}_1$.
  - Assume that we can find the **estimated standard error** se_d of the difference and that this has degrees of freedom df.
  - Assume that the samples either are large or come from a normal population.
- Then we can construct a
  - confidence interval for the unknown population difference of means $\mu_2 - \mu_1$ by

$$(\bar{y}_2 - \bar{y}_1) \pm t_{crit} se_d,$$

    where the critical t-score, t_{crit}, determines the confidence level.
  - significance test:
    - for the null hypothesis $H_0: \ \mu_2 - \mu_1 = 0$ and alternative hypothesis $H_a: \ \mu_2 - \mu_1 \neq 0$.
    - which uses the test statistic: $t_{obs} = \frac{(\bar{y}_2 - \bar{y}_1) - 0}{se_d}$, that has to be evaluated in a t-distribution with df degrees of freedom.

1.15 Comparison of two means (Independent samples)

- In the independent samples situation it can be shown that

$$se_d = \sqrt{se_1^2 + se_2^2},$$

  where se_1 and se_2 are estimated standard errors for the sample means in populations 1 and 2, respectively.
- We recall, that for these we have $se = \frac{s}{\sqrt{n}}$, i.e.

$$se_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},$$

  where s_1 and s_2 are estimated standard deviations for population 1 and 2, respectively.
- **The degrees of freedom** df for se_d can be estimated by a complicated formula, which we will not present here.
- For the confidence interval and the significance test we note that:
  - If both n_1 and n_2 are above 30, then we can use the standard normal distribution (z-score) rather than the t-distribution (t-score).
  - If n_1 or n_2 are below 30, then we let **R** calculate the degrees of freedom and p-value/confidence interval.

1.16 Example: Comparing two means (independent samples)

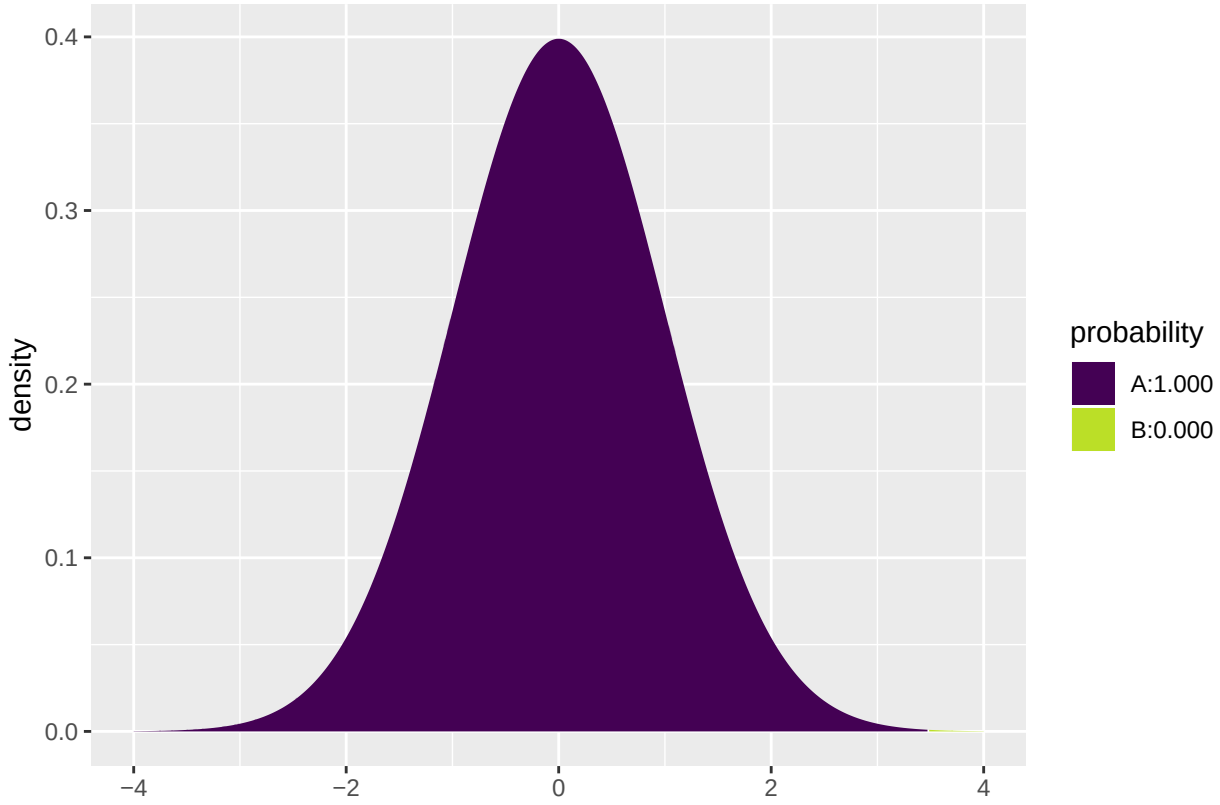
We return to the `Chile` data. We study the association between the variables `sex` and `statusquo` (scale of support for the status-quo). So, we will perform a significance test to test for difference in the mean of `statusquo` for male and females.

```
Chile <- read.delim("https://asta.math.aau.dk/datasets?file=Chile.txt")
library(mosaic)
fv <- favstats(statusquo ~ sex, data = Chile)
fv
```

```
##   sex   min     Q1 median    Q3  max    mean    sd    n missing
## 1   F -1.80 -0.975  0.121 1.033 2.02  0.0657 1.003 1368      11
## 2   M -1.74 -1.032 -0.216 0.861 2.05 -0.0684 0.993 1315       6
```

- Difference: $d = 0.0657 - (-0.0684) = 0.1341$.
- Estimated standard deviations: $s_1 = 1.0032$ (females) and $s_2 = 0.9928$ (males).
- Sample sizes: $n_1 = 1368$ and $n_2 = 1315$.
- Estimated standard error of difference: $se_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{1.0032^2}{1368} + \frac{0.9928^2}{1315}} = 0.0385$.
- Observed t-score for $H_0: \mu_1 - \mu_2 = 0$ is: $t_{obs} = \frac{d-0}{se_d} = \frac{0.1341}{0.0385} = 3.4786$.
- Since both sample sizes are "pretty large" (> 30), we can use the z-score instead of the t-score for finding the p-value (i.e. we use the standard normal distribution):

```
1 - pdist("norm", q = 3.4786, xlim = c(-4, 4))
```




```
## [1] 0.0002520202
```

- Then the p-value is $2 \cdot 0.00025 = 0.0005$, so we reject the null hypothesis.
- We can leave all the calculations to **R** by using `t.test`:

```
t.test(statusquo ~ sex, data = Chile)
```

```
##
##  Welch Two Sample t-test
##
## data:  statusquo by sex
```

```
## t = 3.4786, df = 2678.7, p-value = 0.0005121
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  0.05849179 0.20962982
## sample estimates:
## mean in group F mean in group M
##      0.06570627     -0.06835453
```

- We recognize the t-score 3.4786 and the p-value 0.0005. The estimated degrees of freedom $df = 2679$ is so large that we can not tell the difference between results obtained using z-score and t-score.

1.17 Comparison of two means: confidence interval (independent samples)

- We have already found all the ingredients to construct a **confidence interval for** $\mu_2 - \mu_1$:
  - $d = \bar{y}_2 - \bar{y}_1$ estimates $\mu_2 - \mu_1$.
  - $se_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$ estimates the standard error of d.
- Then:

$$d \pm t_{crit} se_d$$

  is a confidence interval for $\mu_2 - \mu_1$.
- The critical t-score, t_{crit} is chosen corresponding to the wanted confidence level. If n_1 and n_2 both are greater than 30, then $t_{crit} = 2$ yields a confidence level of approximately 95%.

1.18 Comparison of two means: paired t-test (dependent samples)

- Experiment:
  - You choose 32 students at random and measure their average reaction time in a driving simulator while they are listening to radio or audio books.
  - Later the same 32 students redo the simulated driving while talking on a cell phone.
- It is interesting to investigate whether or not the fact that you are actively participating in a conversation changes your average reaction time compared to when you are passively listening.
- So we have 2 samples corresponding to with/without phone. In this case we have **dependent** samples, since we have 2 measurement for each student.
- We use the following strategy for analysis:
  - For each student calculate **the change** in average reaction time with and without talking on the phone.
  - The changes $d_1, d_2, \ldots, d_{32}$ are now considered as **ONE** sample from a population with mean μ.
  - Test the hypothesis $H_0 : \mu = 0$ as usual (using a t-test for testing the mean as in the previous lecture).

1.18.1 Reaction time example

- Data is organized in a data frame with 3 variables:
  - `student` (integer – a simple id)
  - `reaction_time` (numeric – average reaction time in milliseconds)
  - `phone` (factor – `yes`/`no` indicating whether speaking on the phone)

```
reaction <- read.delim("https://asta.math.aau.dk/datasets?file=reaction.txt")
head(reaction, n = 3)
```

```
##   student reaction_time phone
## 1       1           604    no
## 2       2           556    no
## 3       3           540    no
```

Instead of doing manual calculations we let **R** perform the significance test (using `t.test` with `paired = TRUE` as our samples are paired/dependent):

```
t.test(reaction_time ~ phone, data = reaction, paired = TRUE)
```

```
##
##  Paired t-test
##
## data:  reaction_time by phone
## t = -5.4563, df = 31, p-value = 5.803e-06
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  -69.54814 -31.70186
## sample estimates:
## mean of the differences
##                  -50.625
```

- With a p-value of 0.0000058 we reject that speaking on the phone has no influence on the reaction time.
- To understand what is going on, we can manually find the reaction time difference for each student and do a one sample t-test on this difference:

```
yes <- subset(reaction, phone == "yes")
no  <- subset(reaction, phone == "no")
reaction_diff <- data.frame(student = no$student, yes = yes$reaction_time, no = no$reaction_time)
reaction_diff$diff <- reaction_diff$yes - reaction_diff$no
head(reaction_diff)
```

```
##   student yes  no diff
## 1       1 636 604   32
## 2       2 623 556   67
## 3       3 615 540   75
## 4       4 672 522  150
## 5       5 601 459  142
## 6       6 600 544   56
```

```
t.test( ~ diff, data = reaction_diff)
```

```
##
##  One Sample t-test
##
## data:  diff
## t = 5.4563, df = 31, p-value = 5.803e-06
```

```
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  31.70186 69.54814
## sample estimates:
## mean of x
##    50.625
```

2 Comparison of two proportions

2.1 Comparison of two proportions

- We consider the situation, where we have two qualitative samples and we investigate whether a given property is present or not:
  - Let the proportion of population 1 which has the property be π_1, which is estimated by $\hat{\pi}_1$ based on a sample of size n_1.
  - Let the proportion of population 2 which has the property be π_2, which is estimated by $\hat{\pi}_2$ based on a sample of size n_2.
  - We are interested in the difference $\pi_2 - \pi_1$, which is estimated by $d = \hat{\pi}_2 - \hat{\pi}_1$.
  - Assume that we can find the **estimated standard error** se_d of the difference.
- Then we can construct
  - an approximate confidence interval for the difference, $\pi_2 - \pi_1$.
  - a significance test.

2.2 Comparison of two proportions: Independent samples

- In the situation where we have independent samples we know that

$$se_d = \sqrt{se_1^2 + se_2^2},$$

  where se_1 and se_2 are the estimated standard errors for the sample proportion in population 1 and 2, respectively.
- We recall, that these are given by $se = \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$, i.e.

$$se_d = \sqrt{\frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_2}}.$$

- A (approximate) confidence interval for $\pi_2 - \pi_1$ is obtained by the usual construction:

$$(\hat{\pi}_2 - \hat{\pi}_1) \pm z_{crit} se_d,$$

  where the critical z-score determines the confidence level.

2.3 Approximate test for comparing two proportions (independent samples)

- We consider the null hypothesis H_0: $\pi_1 = \pi_2$ (equivalently $H_0 : \pi_1 - \pi_2 = 0$) and the alternative hypothesis H_a: $\pi_1 \neq \pi_2$.

- Assuming H_0 is true, we have a common proportion π, which is estimated by

$$\hat{\pi} = \frac{n_1\hat{\pi}_1 + n_2\hat{\pi}_2}{n_1 + n_2},$$

  i.e. we aggregate the populations and calculate the relative frequency of the property (with other words: we estimate the proportion, π, as if the two samples were one).
- Rather than using the estimated standard error of the difference from previous, we use the following that holds under H_0:

$$se_0 = \sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

- The observed test statistic/z-score for H_0 is then:

$$z_{obs} = \frac{(\hat{\pi}_2 - \hat{\pi}_1) - 0}{se_0},$$

  which is evaluated in the standard normal distribution.
- The p-value is calculated in the usual way.

WARNING: The approximation is only good, when $n_1\hat{\pi},\ n_1(1-\hat{\pi}),\ n_2\hat{\pi},\ n_2(1-\hat{\pi})$ all are greater than 5.

2.4 Example: Approximate confidence interval and test for comparing proportions

We return to the `Chile` dataset. We make a new binary variable indicating whether the person intends to vote no or something else (and we remember to tell **R** that it should think of this as a grouping variable, i.e. a `factor`):

```
Chile$voteNo <- relevel(factor(Chile$vote == "N"), ref = "TRUE")
```

We study the association between the variables `sex` and `voteNo`:

```
tab <- tally( ~ sex + voteNo, data = Chile, useNA = "no")
tab
```

```
##    voteNo
## sex TRUE FALSE
##   F  363   946
##   M  526   697
```

This gives us all the ingredients needed in the hypothesis test:

- Estimated proportion of men that vote no: $\hat{\pi}_1 = \frac{526}{526+697} = 0.430$
- Estimated proportion of women that vote no: $\hat{\pi}_2 = \frac{363}{363+946} = 0.277$

2.5 Example: Approximate confidence interval (cont.)

- Estimated difference:

$$d = \hat{\pi}_2 - \hat{\pi}_1 = 0.277 - 0.430 = -0.153$$

- Standard error of difference:

$$se_d = \sqrt{\frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_2}}$$
$$= \sqrt{\frac{0.430(1-0.430)}{1223} + \frac{0.277(1-0.277)}{1309}} = 0.0188.$$

- Approximate 95% confidence interval for difference:

$$d \pm 1.96 \cdot se_d = (-0.190, -0.116).$$

2.6 Example: p-value (cont.)

- Estimated common proportion:

$$\hat{\pi} = \frac{1223 \times 0.430 + 1309 \times 0.277}{1309 + 1223} = \frac{526 + 363}{1309 + 1223} = 0.351.$$

- Standard error of difference when $H_0: \pi_1 = \pi_2$ is true:

$$se_0 = \sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = 0.0190.$$

- The observed test statistic/z-score:

$$z_{obs} = \frac{d}{se_0} = -8.06.$$

- The test for H_0 against $H_a: \pi_1 \neq \pi_2$ yields a p-value that is practically zero, i.e. we can reject that the proportions are equal.

2.7 Automatic calculation in R

```
Chile2 <- subset(Chile, !is.na(voteNo))
prop.test(voteNo ~ sex, data = Chile2, correct = FALSE)
```

```
##
##  2-sample test for equality of proportions without continuity
##  correction
##
## data:  tally(voteNo ~ sex)
## X-squared = 64.777, df = 1, p-value = 8.389e-16
## alternative hypothesis: two.sided
## 95 percent confidence interval:
##  -0.1896305 -0.1159275
## sample estimates:
##    prop 1    prop 2
## 0.2773109 0.4300899
```

2.8 Fisher's exact test

- If $n_1\hat{\pi},\ n_1(1-\hat{\pi}),\ n_2\hat{\pi},\ n_2(1-\hat{\pi})$ are not all greater than 5, then the approximate test cannot be trusted. Instead you can use Fisher's exact test:

```
fisher.test(tab)
```

```
##
##  Fisher's Exact Test for Count Data
##
## data:  tab
## p-value = 1.04e-15
## alternative hypothesis: true odds ratio is not equal to 1
## 95 percent confidence interval:
##  0.4292768 0.6021525
## sample estimates:
## odds ratio
##  0.5085996
```

- Again the p-value is seen to be extremely small, so we definitely reject the null hypothesis of equal `voteNo` proportions for women and men.

2.9 Agresti: Overview of comparison of two groups

TABLE 7.10: Summary of Comparison Methods for Two Groups, for Independent Random Samples

		Type of Response Variable	
		Categorical	Quantitative
Estimation			
1.	Parameter	$\pi_2 - \pi_1$	$\mu_2 - \mu_1$
2.	Point estimate	$\hat{\pi}_2 - \hat{\pi}_1$	$\bar{y}_2 - \bar{y}_1$
3.	Standard error	$se = \sqrt{\frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_2}}$	$se = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$
4.	Confidence interval	$(\hat{\pi}_2 - \hat{\pi}_1) \pm z(se)$	$(\bar{y}_2 - \bar{y}_1) \pm t(se)$
Significance testing			
1.	Assumptions	Randomization ≥10 observations in each category, for each group	Randomization Normal population dist.'s (robust, especially for large n's)
2.	Hypotheses	H_0: $\pi_1 = \pi_2$ $(\pi_2 - \pi_1 = 0)$ H_a: $\pi_1 \neq \pi_2$	H_0: $\mu_1 = \mu_2$ $(\mu_2 - \mu_1 = 0)$ H_a: $\mu_1 \neq \mu_2$
3.	Test statistic	$z = \frac{\hat{\pi}_2 - \hat{\pi}_1}{se_0}$	$t = \frac{\bar{y}_2 - \bar{y}_1}{se}$
4.	P-value	Two-tail probability from standard normal or t (Use one tail for one-sided alternative)	

3 Contingency tables

3.1 A contingency table

- The dataset `popularKids`, we study the **association** between the **factors** `Goals` and `Urban.Rural`:
  - `Urban.Rural`: The students were selected from urban, suburban, and rural schools.
  - `Goals`: The students indicated whether good grades, athletic ability, or popularity was most important to them.
  - In total 478 students from grades 4-6.
- Based on a sample we make a cross tabulation of the factors and we get a so-called **contingency table** (krydstabel).

```
popKids <- read.delim("https://asta.math.aau.dk/datasets?file=PopularKids.txt")
library(mosaic)
tab <- tally(~Urban.Rural + Goals, data = popKids, margins = TRUE)
tab
```

```
##            Goals
## Urban.Rural Grades Popular Sports Total
##    Rural        57      50     42   149
##    Suburban     87      42     22   151
##    Urban       103      49     26   178
##    Total       247     141     90   478
```

3.2 A conditional distribution

- Another representation of data is the probability distribution of `Goals` for each level of `Urban.Rural`, i.e. the sum in each row of the table is 1 (up to rounding):

```
##            Goals
## Urban.Rural Grades Popular Sports   Sum
##    Rural     0.383   0.336  0.282 1.000
##    Suburban  0.576   0.278  0.146 1.000
##    Urban     0.579   0.275  0.146 1.000
##    Total     0.517   0.295  0.188 1.000
```

- Here we will talk about the **conditional distribution** of `Goals` given `Urban.Rural`.
- An important question could be:
  - Are the goals of the kids different when they come from urban, suburban or rural areas? I.e. are the rows in the table significantly different?
- There is (almost) no difference between urban and suburban, but it looks like rural is different.

3.3 Independence

- Recall, that two factors are **independent**, when there is no difference between the population's distributions of one factor given the levels of the other factor.
- Otherwise the factors are said to be **dependent**.
- If we e.g. have the following conditional **population distributions** of `Goals` given `Urban.Rural`:

```
##            Goals
## Urban.Rural Grades Popular Sports
##    Rural       0.5     0.3    0.2
##    Suburban    0.5     0.3    0.2
##    Urban       0.5     0.3    0.2
```

- Then the factors `Goals` and `Urban.Rural` are independent.
- We take a sample and "measure" the factors F_1 and F_2. E.g. `Goals` and `Urban.Rural` for a random child.
- The hypothesis of interest today is:

$$H_0 : F_1 \text{ and } F_2 \text{ are independent}, \quad H_a : F_1 \text{ and } F_2 \text{ are dependent.}$$

3.4 The Chi-squared test for independence

- Our best guess of the distribution of `Goals` is the relative frequencies in the sample:

```
tab <- tally(~Urban.Rural + Goals, data = popKids)
n <- margin.table(tab)
pctGoals <- round(margin.table(tab, 2) / n, 3)
pctGoals
```

```
## Goals
##  Grades Popular  Sports
##   0.517   0.295   0.188
```

- If we assume independence, then this is also a guess of the conditional distributions of `Goals` given `Urban.Rural`.
- The corresponding expected counts in the sample are then:

```
##            Goals
## Urban.Rural Grades         Popular       Sports        Sum
##    Rural     77.0 (0.517)  44.0 (0.295)  28.1 (0.188) 149.0 (1.000)
##    Suburban  78.0 (0.517)  44.5 (0.295)  28.4 (0.188) 151.0 (1.000)
##    Urban     92.0 (0.517)  52.5 (0.295)  33.5 (0.188) 178.0 (1.000)
##    Sum      247.0 (0.517) 141.0 (0.295)  90.0 (0.188) 478.0 (1.000)
```

3.5 Calculation of expected table

```
pctexptab
```

```
##            Goals
## Urban.Rural Grades        Popular       Sports        Sum
##    Rural     77.0 (0.517)  44.0 (0.295)  28.1 (0.188) 149.0 (1.000)
##    Suburban  78.0 (0.517)  44.5 (0.295)  28.4 (0.188) 151.0 (1.000)
##    Urban     92.0 (0.517)  52.5 (0.295)  33.5 (0.188) 178.0 (1.000)
##    Sum      247.0 (0.517) 141.0 (0.295)  90.0 (0.188) 478.0 (1.000)
```

- We note that
  - The relative frequency for a given column is **column total** divided by **table total**. For example `Grades`, which is $\frac{247}{478} = 0.517$.
  - The expected value in a given cell in the table is then the cell's relative column frequency multiplied by the cell's **row total**. For example `Rural` and `Grades`: $149 \times 0.517 = 77.0$.
- This can be summarized to:
  - The expected value in a cell is the product of the cell's **row total** and **column total** divided by the **table total**

3.6 Chi-squared (χ^2) test statistic

- We have an **observed table**:

```
tab
```

```
##            Goals
## Urban.Rural Grades Popular Sports
##    Rural        57      50     42
##    Suburban     87      42     22
##    Urban       103      49     26
```

- And an **expected table**, if H_0 is true:

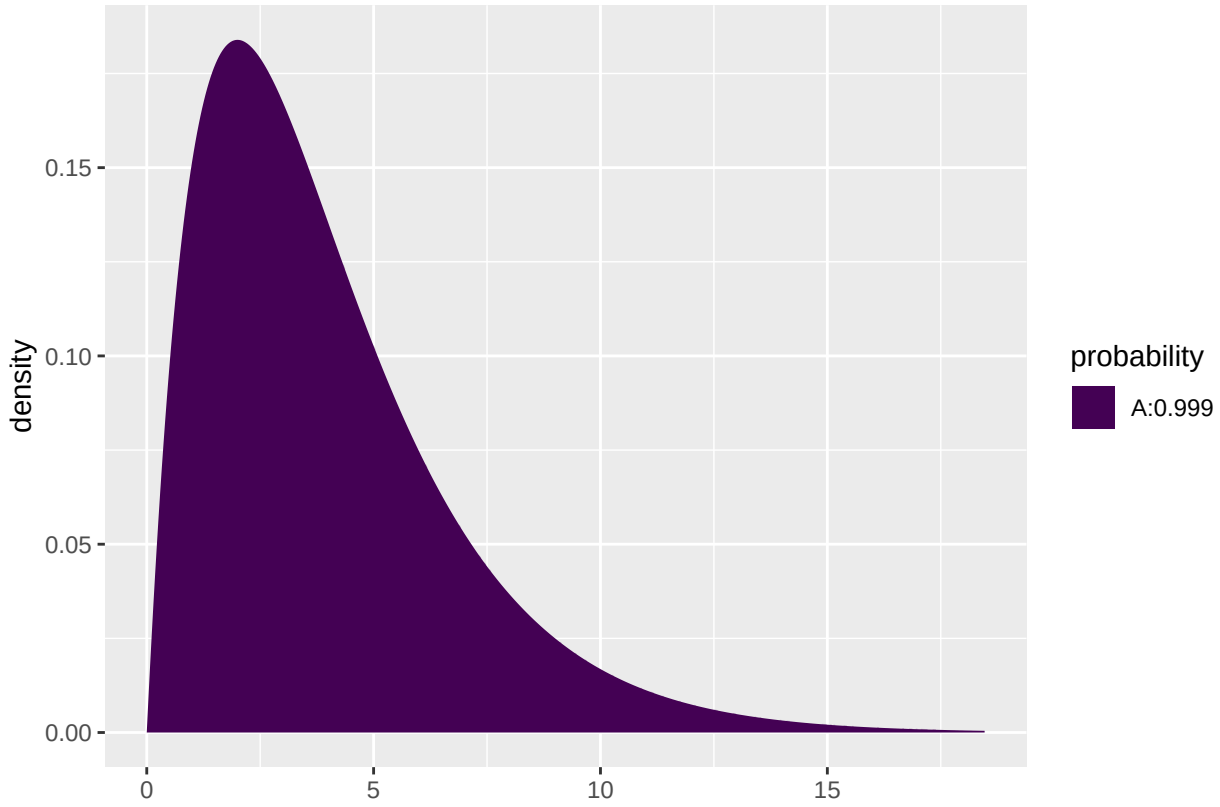
```
##            Goals
## Urban.Rural Grades Popular Sports Sum
##    Rural     77.0   44.0    28.1  149.0
##    Suburban  78.0   44.5    28.4  151.0
##    Urban     92.0   52.5    33.5  178.0
##    Sum      247.0  141.0    90.0  478.0
```

- If these tables are "far from each other", then we reject H_0. We want to measure the distance via the Chi-squared test statistic:
  - $X^2 = \sum \frac{(f_o - f_e)^2}{f_e}$: Sum over all cells in the table
  - f_o is the frequency in a cell in the observed table
  - f_e is the corresponding frequency in the expected table.
- We have:
$$X^2_{obs} = \frac{(57-77)^2}{77} + \ldots + \frac{(26-33.5)^2}{33.5} = 18.8$$
- Is this a large distance??

3.7 χ^2-test template.

- We want to test the hypothesis H_0 of independence in a table with r rows and c columns:
  - We take a sample and calculate X^2_{obs} - the observed value of the test statistic.
  - p-value: Assume H_0 is true. What is then the chance of obtaining a larger X^2 than X^2_{obs}, if we repeat the experiment?
- This can be approximated by the **χ^2-distribution** with $df = (r-1)(c-1)$ degrees of freedom.
- For `Goals` and `Urban.Rural` we have $r = c = 3$, i.e. $df = 4$ and $X^2_{obs} = 18.8$, so the p-value is:

```
1 - pdist("chisq", 18.8, df = 4)
```




```
## [1] 0.0008603303
```

- There is clearly a significant association between `Goals` and `Urban.Rural`.

3.8 The function `chisq.test`

- All of the above calculations can be obtained by the function `chisq.test`.

```
tab <- tally(~ Urban.Rural + Goals, data = popKids)
testStat <- chisq.test(tab, correct = FALSE)
testStat
```

```
##
##  Pearson's Chi-squared test
##
## data:  tab
## X-squared = 18.828, df = 4, p-value = 0.0008497
```

```
testStat$expected
```

```
##            Goals
## Urban.Rural   Grades  Popular   Sports
##    Rural    76.99372 43.95188 28.05439
##    Suburban 78.02720 44.54184 28.43096
##    Urban    91.97908 52.50628 33.51464
```

- The frequency data can also be put directly into a matrix.

```
data <- c(57, 87, 103, 50, 42, 49, 42, 22, 26)
tab <- matrix(data, nrow = 3, ncol = 3)
row.names(tab) <- c("Rural", "Suburban", "Urban")
colnames(tab) <- c("Grades", "Popular", "Sports")
tab
```

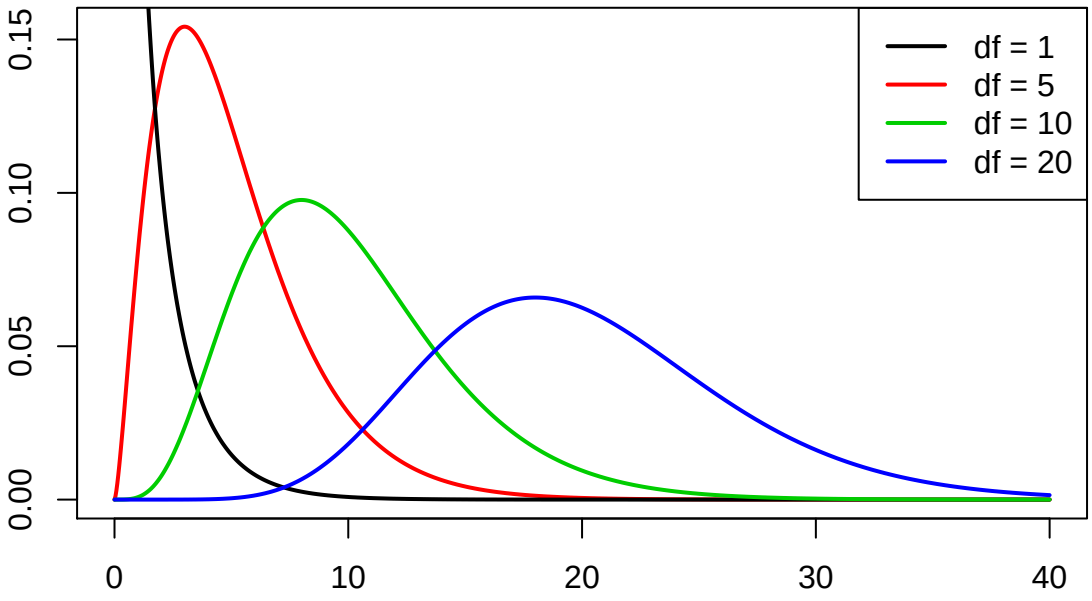
```
##          Grades Popular Sports
## Rural        57      50     42
## Suburban     87      42     22
## Urban       103      49     26
```

```
chisq.test(tab)
```

```
##
##  Pearson's Chi-squared test
##
## data:  tab
## X-squared = 18.828, df = 4, p-value = 0.0008497
```

3.9 The χ^2-distribution

- The χ^2-distribution with df degrees of freedom:
  - Is never negative. And $X^2 = 0$ only happens if $f_e = f_o$.
  - Has mean $\mu = df$
  - Has standard deviation $\sigma = \sqrt{2df}$
  - Is skewed to the right, but approaches a normal distribution when df grows.



3.10 Summary

- For the the Chi-squared statistic, X^2, to be appropriate we require that the expected values have to be $f_e \geq 5$.
- Now we can summarize the ingredients in the Chi-squared test for independence.

TABLE 8.5: The Five Parts of the Chi-Squared Test of Independence

1. Assumptions: Two categorical variables, random sampling, $f_e \geq 5$ in all cells
2. Hypotheses: H_0: Statistical independence of variables
   H_a: Statistical dependence of variables
3. Test statistic: $\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e}$, where $f_e = \frac{(\text{Row total})(\text{Column total})}{\text{Total sample size}}$
4. P-value: P = right-tail probability above observed χ^2 value,
   for chi-squared distribution with $df = (r-1)(c-1)$
5. Conclusion: Report P-value
   If decision needed, reject H_0 at α-level if $P \leq \alpha$

3.11 Residual analysis

- If we reject the hypothesis of independence it can be of interest to identify the significant deviations.
- In a given cell in the table, $f_o - f_e$ is the deviation between data and the expected values under the null hypothesis.
- We assume that $f_e \geq 5$.
- If H_0 is true, then the standard error of $f_o - f_e$ is given by

$$se = \sqrt{f_e(1 - \textbf{row proportion})(1 - \textbf{column proportion})}$$

- The corresponding z-score
$$z = \frac{f_o - f_e}{se}$$
should in 95% of the cells be between ± 2. Values above 3 or below -3 should not appear.
- In `popKids` table cell `Rural and Grade` we got $f_e = 77.0$ and $f_o = 57$. Here **column proportion** $= 0.517$ and **row proportion** $= 149/478 = 0.312$.
- We can then calculate
$$z = \frac{57 - 77}{\sqrt{77(1 - 0.517)(1 - 0.312)}} = -3.95$$
- Compared to the null hypothesis there are way too few rural kids who find grades important.
- In summary: The standardized residuals allow for cell-by-cell (f_e vs f_o) comparision.

3.12 Residual analysis in R

- In R we can extract the standardized residuals from the output of `chisq.test`:

```
tab <- tally(~ Urban.Rural + Goals, data = popKids)
testStat <- chisq.test(tab, correct = FALSE)
testStat$stdres
```

```
##            Goals
## Urban.Rural     Grades    Popular     Sports
##    Rural    -3.9508449  1.3096235  3.5225004
##    Suburban  1.7666608 -0.5484075 -1.6185210
##    Urban     2.0865780 -0.7274327 -1.8186224
```

3.13 Cramér's V

- To measure the strength of the association, the Swedish mathematician Harald Cramér developed a measure which is estimated by
$$V = \sqrt{\frac{X^2}{n \cdot \min(r - 1, c - 1)}}$$
where r and c are the number of columns and rows in the contingency table and n is the sample size.
- Property:
  - Cramér's V lies between 0(no association) and 1(complete association)
- In the situation with the factors `Goals` and `Urban.Rural` from the dataset `popularKids` we get
$$V = \sqrt{\frac{X^2}{n \cdot \min(r - 1, c - 1)}} = \sqrt{\frac{18.8}{479 \cdot \min(3 - 1, 3 - 1)}} = 0.14,$$
which indicates a weak (but significant) association.
- The function `CramerV` in the package `DescTools` gives you the value and a confidence interval

```r
library(DescTools)
```

```
##
## Attaching package: 'DescTools'

## The following object is masked from 'package:mosaic':
##
##     MAD
```

```r
CramerV(tab, conf = 0.95, type = "perc")
```

```
##   Cramer V     lwr.ci     upr.ci
## 0.14033592 0.06014641 0.19419139
```

4 Ordinal variables

4.1 Association between ordinal variables

- For a random sample of black males the General Social Survey in 1996 asked two questions:
  - Q1: What is your yearly income (`income`)?
  - Q2: How satisfied are you with your job (`satisfaction`)?
- Both measurements are on an ordinal scale.

	VeryD	LittleD	ModerateS	VeryS
< 15k	1	3	10	6
15-25k	2	3	10	7
25-40k	1	6	14	12
> 40k	0	1	9	11

- We might do a chi-square test to see whether Q1 and Q2 are associated, but the test does not exploit the ordinality.
- We shall consider a test that incorporates ordinality.

4.2 Gamma coefficient

- Consider a pair of respondents, where **respondent 1** is below **respondent 2** in relation to Q1.
  - If **respondent 1** is also below **respondent 2** in relation to Q2 then the pair is *concordant*.
  - If **respondent 1** is above **respondent 2** in relation to Q2 then the pair is *disconcordant*.
- Let:

C = the number of concordant pairs in our sample.

D = the number of disconcordant pairs in our sample.

- We define the estimated *gamma coefficient*

$$\hat{\gamma} = \frac{C-D}{C+D} = \underbrace{\frac{C}{C+D}}_{concordant\ prop.} - \underbrace{\frac{D}{C+D}}_{discordant\ prop.}$$

4.3 Gamma coefficient

- Properties:
  - Gamma lies between -1 og 1
  - The sign tells whether the association is positive or negative
  - Large absolute values correspond to strong association
- The standard error $se(\hat{\gamma})$ on $\hat{\gamma}$ is complicated to calculate, so we leave that to software.
- We can now determine a 95% confidence interval:

$$\hat{\gamma} \pm 1.96 se(\hat{\gamma})$$

  and if zero is contained in the interval, then there is no significant association, when we perform a test with a 5% significance level.

4.4 Example

- First, we need to install the package `vcdExtra`, which has the function `GKgamma` for calculating gamma. It also has the dataset on job satisfaction and income built-in:

```
library(vcdExtra)
JobSat
```

```
##         satisfaction
## income   VeryD LittleD ModerateS VeryS
##   < 15k      1       3        10     6
##   15-25k     2       3        10     7
##   25-40k     1       6        14    12
##   > 40k      0       1         9    11
```

```
GKgamma(JobSat, level = 0.90)
```

```
## gamma        : 0.221
## std. error   : 0.117
## CI           : 0.028 0.414
```

- A positive association. Marginally significant at the 10% level, but not so at the 5% level.

5 Validation of data

5.1 Goodness of fit test

- You have collected a sample and want to know, whether the sample is representative for people living in Hirtshals.
- E.g. whether the distribution of gender, age, or profession in the sample do not differ significantly from the distribution in Hirtshals.
- Actually, you know how to do that for binary variables like gender, but not if you e.g. have 6 agegroups.

5.2 Example

- As an example we look at k groups, where data from Hjørring kommune tells us the distribution in Hirtshals is given by the vector
$$\pi = (\pi_1, \ldots, \pi_k),$$
where π_i is the proportion which belongs to group number i, $\quad i = 1, 2 \ldots, k$ in Hirtshals.
- Consider the sample represented by the vector:
$$O = (O_1, \ldots, O_k),$$
where O_i is the observed number of individuals in group number i, $\quad i = 1, 2, \ldots, k$.
- The total number of individuals:
$$n = \sum_{i=1}^{k} O_i.$$
- The expected number of individuals in each group, if we have a sample from Hirtshals:
$$E_i = n\pi_i, \ i = 1, 2, \ldots, k$$

5.3 Goodness of fit test

- We will use the following measure to see how far away the observed is from the expected:
$$X^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$
- If this is large we reject the hypothesis that the sample has the same distribution as Hirtshals. The reference distribution is the χ^2 with $k - 1$ degrees of freedom.

5.4 Example

- Assume we have four groups and that the true distribution is given by:

```
k <- 4
pi_vector <- c(0.3, 0.2, 0.25, 0.25)
```

- Assume that we have the following sample:

```
O_vector <- c(74, 72, 40, 61)
```

- Expected number of individuals in each group:

```
n <- sum(O_vector)
E_vector <- n * pi_vector
E_vector
```

```
## [1] 74.10 49.40 61.75 61.75
```

- X^2 statistic:

```
Xsq = sum((O_vector - E_vector)^2 / E_vector)
Xsq
```

```
## [1] 18.00945
```

- p-value:

```
p_value <- 1 - pchisq(Xsq, df = k-1)
p_value
```

```
## [1] 0.0004378808
```

5.5 Test in R

```
Xsq_test <- chisq.test(O_vector, p = pi_vector)
Xsq_test
```

```
##
##  Chi-squared test for given probabilities
##
## data:  O_vector
## X-squared = 18.009, df = 3, p-value = 0.0004379
```

- As the hypothesis is rejected, we look at the standardized residuals (z-scores):

```
Xsq_test$stdres
```

```
## [1] -0.01388487  3.59500891 -3.19602486 -0.11020775
```

- We conclude that group 1 and 4 is close to true distribution in Hirtshals, but in groups 2 og 3 we have a significant mismatch.