

# Analysis of Variance

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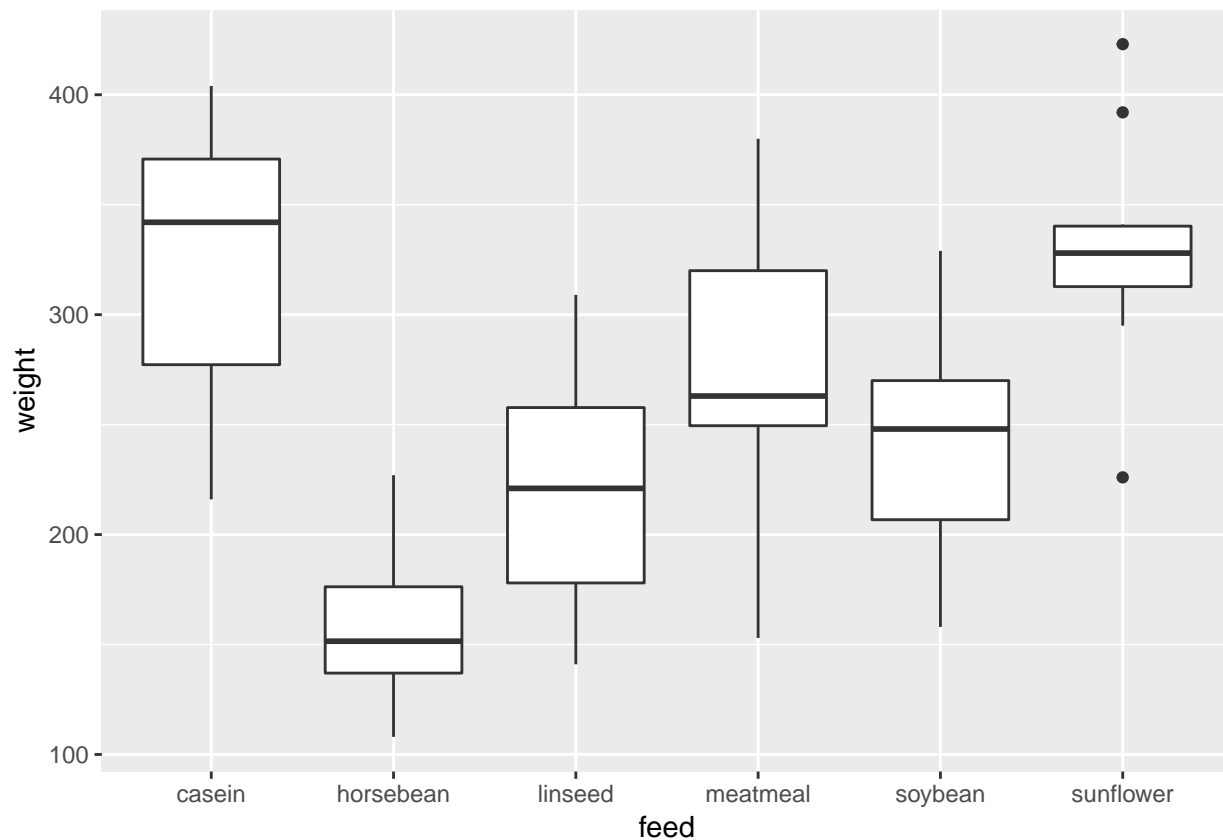
## 1 One way analysis of variance

### 1.1 Example

- The data set `chickwts` is available in  $\mathbf{R}$ , and on the course webpage.

- 71 newly hatched chicks were randomly allocated into six groups, and each group was given a different feed supplement.
- Their weights in grams after six weeks are given along with feed types, i.e. we have a sample with corresponding measurements of 2 variables:
  - `weight`: a numeric variable giving the chick weight.
  - `feed`: a factor giving the feed type.
- Always start with some graphics:

```
library(mosaic)
gf_boxplot(weight ~ feed, data = chickwts)
```



## 1.2 The ANOVA Model

- We measure the response  $y$  which in this case is `weight`.
- We want to study the effect of the factor  $x$  on  $y$ . In this case  $x = \text{feed}$  and divides the sample in  $g = 6$  groups.
- The mean responses within the groups are denoted  $\mu_1, \mu_2, \dots, \mu_g$ .
- We will assume that
  - $y = \mu_x + \epsilon$ , when  $y$  is a response in group  $x$
  - $\epsilon$  are a sample from a population with mean zero and standard deviation  $\sigma$ .
  - The standard deviation for the population in each group is the same and equals  $\sigma$
  - The response variable,  $y$ , is normal distributed within each group.
- The ANOVA test is a *test of equal means* for the different groups.

## 2 Estimation of mean values

### 2.1 Estimates

- Least squares estimates for population means  $\hat{\mu}_x$  is given by the average of the response measurements in group  $x$ .
- For a given measured response  $y$  we let  $\hat{y}$  denote the model's prediction of  $y$ , i.e.

$$\hat{y} = \hat{\mu}_x$$

if  $y$  is a response for an observation in group  $x$ .

- We use `mean` to find the mean, for each group:

```
mean(weight ~ feed, data = chickwts)
```

```
## casein horsebean linseed meatmeal soybean sunflower
## 323.5833 160.2000 218.7500 276.9091 246.4286 328.9167
```

- We can e.g. see that  $\hat{y} = 323.6$ , when `feed=casein` but  $\hat{y} = 160.2$ , when `feed=horsebean`.
- Is it a significant difference ?

### 2.2 Contrast coding

- In many cases there is a group corresponding to “no treatment” and we are interested in the effect of different treatments.
- In this example we only have different `feeds`, which are sorted in lexicographical order by R, so `casein` is the reference.
- We can specify the model via:
  - `Intercept` corresponding to the mean response for the reference (`casein`).
  - For each of the other groups we have a **contrast**, which measures **the difference** between the mean value for that group and the reference group.
- For a given contrast we can calculate standard error, t-score and p-value, and thereby investigate whether there is a difference between this group and the reference group.
- In Agresti this is referred to as using **dummy variables**.

### 2.3 Example

```
model <- lm(weight ~ feed, data = chickwts)
summary(model)
```

```
##
## Call:
## lm(formula = weight ~ feed, data = chickwts)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -123.909  -34.413   1.571   38.170  103.091
##
## Coefficients:
```

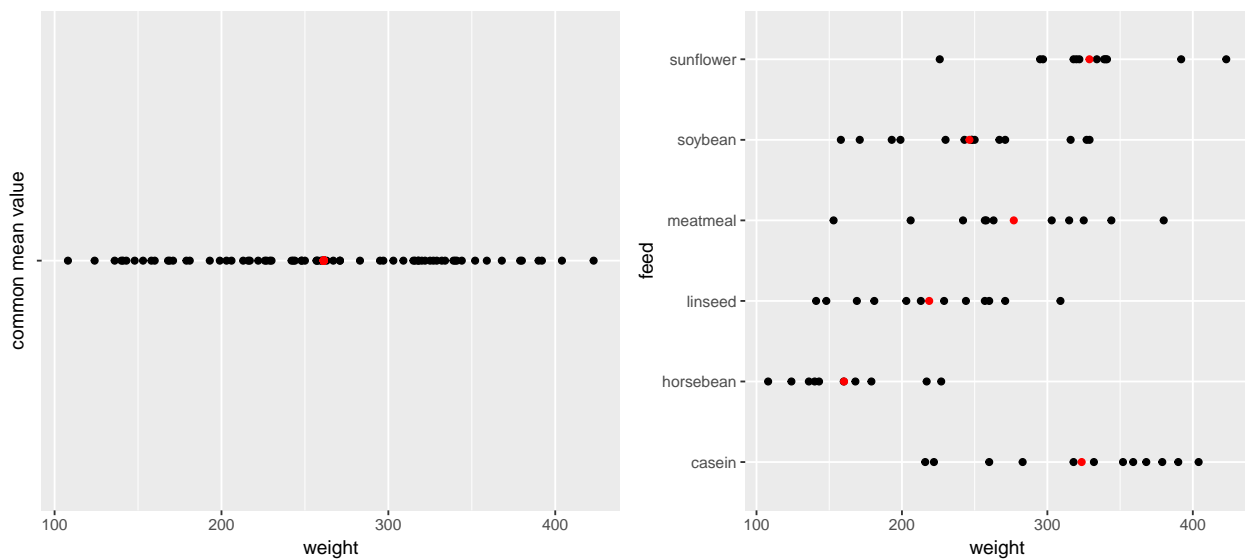
```
##           Estimate Std. Error t value Pr(>|t|)
## (Intercept)   323.583    15.834  20.436 < 2e-16 ***
## feedhorsebean -163.383    23.485  -6.957 2.07e-09 ***
## feedlinseed   -104.833    22.393  -4.682 1.49e-05 ***
## feedmeatmeal  -46.674    22.896  -2.039 0.045567 *
## feedsoybean   -77.155    21.578  -3.576 0.000665 ***
## feedsunflower  5.333     22.393   0.238 0.812495
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 54.85 on 65 degrees of freedom
## Multiple R-squared:  0.5417, Adjusted R-squared:  0.5064
## F-statistic: 15.36 on 5 and 65 DF,  p-value: 5.936e-10
```

- We get information about contrasts and their significance:
- Intercept corresponding to casein has weight different from zero ( $p < 2 \times 10^{-16}$ ) (of course, chickens grow a lot over 6 weeks)
- Weight difference between casein and horsebean is extremely significant ( $p=2 \times 10^{-9}$ ).
- There is no significant weight difference between casein and sunflower ( $p=81\%$ ).

### 3 Overall test for effect

#### 3.1 Graphical representation of models

- We have two alternative explanations of the data.
- Simple model with one parameter (mean): “The feed type doesn’t matter. The weight is just random around a common mean value”.
- Complex model with six parameters (means): “The feed type is important. For each feed type we get a different mean value and the weights are random around these values.”



## 3.2 Hypotheses and test statistic

- Is the complex model significantly better (i.e. is there any effect of the explanatory grouping variable)? We can write the corresponding hypotheses in two different ways

$$H_0 : \mu_1 = \mu_2 = \dots = \mu_g \quad \text{against} \quad H_a : \text{At least 2 of the population means are different}$$

- Alternatively

$$H_0 : \text{All contrasts are equal to zero.} \quad H_a : \text{At least one contrast is non-zero.}$$

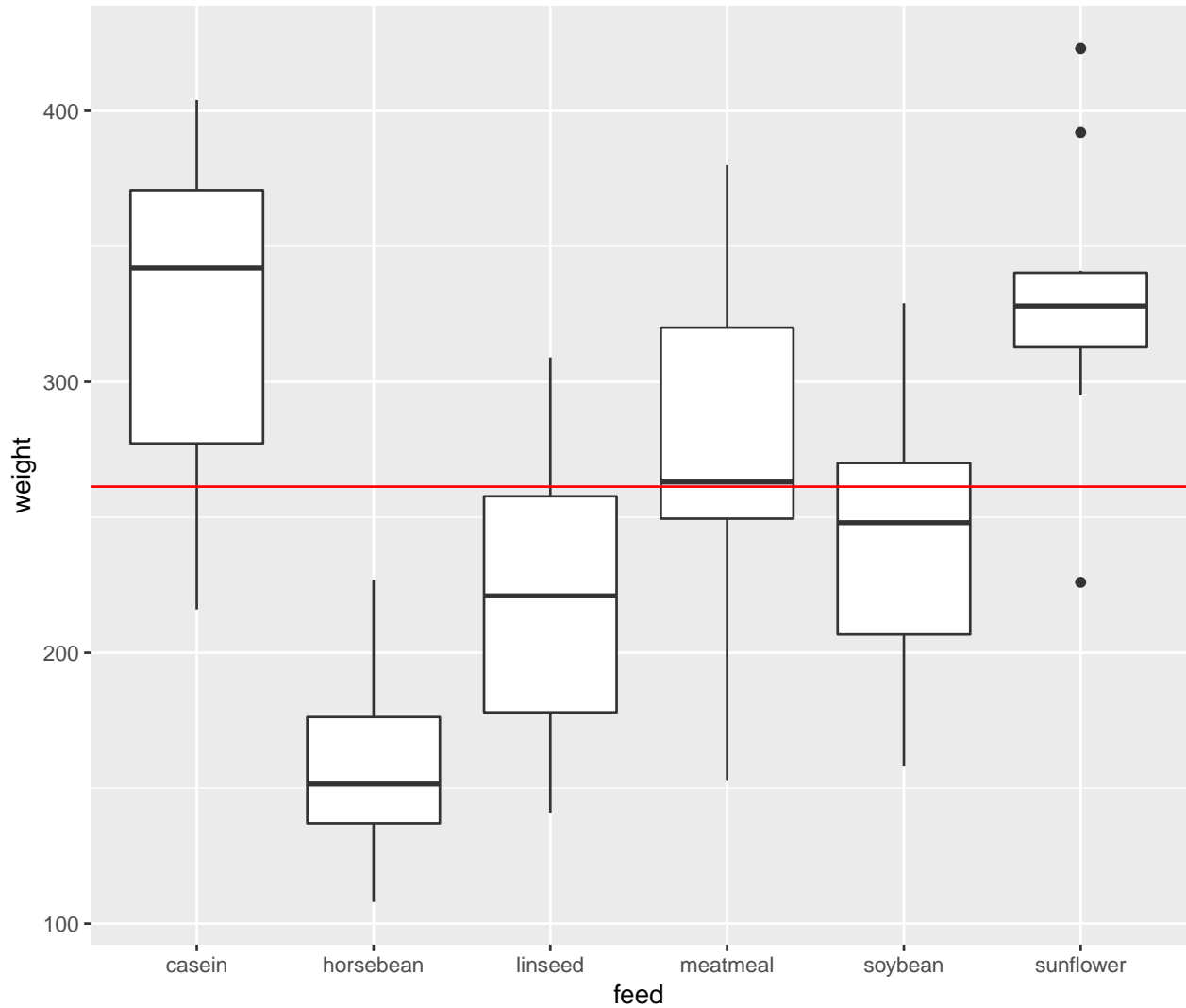
- We will (indirectly) use  $R^2$  to do the test. If it is large, the complex model has good predictive power compared to the simple model. To judge significance we use

$$F_{obs} = \frac{(n-g)R^2}{(g-1)(1-R^2)} = \frac{(TSS - SSE)/(g-1)}{SSE/(n-g)}.$$

- Large values of  $R^2$  implies large values of  $F_{obs}$ , which points to the alternative hypothesis.
- I.e. when we have calculated the observed value  $F_{obs}$ , then we have to find the probability that a new experiment would result in a larger value.
- TSS: error sum of squares if common mean. SSE: error sum of squares if different means.
- TSS-SSE: how much does error sum of squares increase if means are restricted to be equal.
- One can show that TSS-SSE is variation of group means around common mean - **variance between groups**

## 3.3 Interpretation of $F$ statistic - Variance between/within groups

- It can be shown that the numerator of  $F_{obs}$  is a measure of **the variance between the groups**, i.e. how much “boxes” vary around the total average (the red line).
- Likewise it can be shown the denominator of  $F_{obs}$  is a measure for **the variance within groups**, i.e. how “tall” the boxes in the boxplot are.



- The bigger deviations between the red line and the box means relative to the variation within boxes, the less we trust  $H_0$ . This is measured by the F-test statistic, which can be stated as

$$F_{obs} = \frac{\text{variance between groups}}{\text{variance within groups}}$$

### 3.4 Example

```
model <- lm(weight ~ feed, data = chickwts)
summary(model)
```

```
##
## Call:
## lm(formula = weight ~ feed, data = chickwts)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
```

```

## -123.909  -34.413    1.571   38.170  103.091
##
## Coefficients:
##           Estimate Std. Error t value Pr(>|t|)
## (Intercept)   323.583    15.834  20.436 < 2e-16 ***
## feedhorsebean -163.383    23.485  -6.957 2.07e-09 ***
## feedlinseed   -104.833    22.393  -4.682 1.49e-05 ***
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## feedsoybean   -77.155    21.578  -3.576 0.000665 ***
## feedsunflower  5.333    22.393   0.238 0.812495
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 54.85 on 65 degrees of freedom
## Multiple R-squared:  0.5417, Adjusted R-squared:  0.5064
## F-statistic: 15.36 on 5 and 65 DF,  p-value: 5.936e-10

```

- The last line gives us the value of  $F_{obs} = 15.36$  and the corresponding  $p$ -value ( $5.9 \times 10^{-10}$ ). Clearly there is a significant difference between the types of `feed`.

## 4 Two way analysis of variance

### 4.1 Additive effects

- The data set `ToothGrowth` is available in **R** and on the webpage. For more info about this data, use `?ToothGrowth`.
- The data describes the tooth length in guinea pigs where some receive vitamin C treatment and others are given orange juice in different dosage.
- A total of 60 observations on 3 variables.
  - `[,1] len` The tooth length
  - `[,2] supp` The type of the supplement (OJ or VC)
  - `[,3] dose` The dosage (LO, ME, HI)
- We will study the response `len` with the predictors `supp` and `dose`.
- At first we look at the model with additive effects
  - $\text{len} = \mu + \text{"effect of supp"} + \text{"effect of dose"} + \text{error}$
- This is also called the main effects model since it does not contain interaction terms.
- The parameter  $\mu$  corresponds to the `Intercept` and is the mean tooth length in the reference group (supp OJ, dose LO).
- The effect of `supp` is the difference in mean when changing from OJ to VC.
- The effect of `dose` is the difference in mean when changing from LO to either ME or HI.

## 4.2 Dummy coding

- Let us introduce dummy variables:
  - $s_C = 1$  if supp VC and zero otherwise.
  - $d_M = 1$  if dose is ME and zero otherwise.
  - $d_H = 1$  if dose is HI and zero otherwise.
- Then we state the model

$$\text{length} = \mu + \beta_1 s_C + \beta_2 d_M + \beta_3 d_H + \text{error}.$$

- Interpretation:
  - $\mu$  is the expected tooth length when supp is OJ and dose is LO ( $s_C = d_M = d_H = 0$ ).
  - $\beta_1$  is the effect of supplement OJ to VC ( $s_C = 1$ ).
  - $\beta_2$  is the effect of increasing dosage from LO to ME ( $d_M = 1$ ).
  - $\beta_3$  is the effect of increasing dosage from LO to HI ( $d_H = 1$ ).
- As a two-way table:

	<i>LO</i>	<i>ME</i>	<i>HI</i>
<i>OJ</i>	$\mu$	$\mu + \beta_2$	$\mu + \beta_3$
<i>VC</i>	$\mu + \beta_1$	$\mu + \beta_1 + \beta_2$	$\mu + \beta_1 + \beta_3$

## 4.3 Main effect model in R

- The main effects model is fitted by

```
MainEff <- lm(len ~ supp + dose, data = ToothGrowth)
summary(MainEff)
```

```
##
## Call:
## lm(formula = len ~ supp + dose, data = ToothGrowth)
##
## Residuals:
##   Min     1Q Median     3Q    Max
## -7.085 -2.751 -0.800  2.446  9.650
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  12.4550     0.9883  12.603 < 2e-16 ***
## suppVC      -3.7000     0.9883  -3.744 0.000429 ***
## doseME       9.1300     1.2104   7.543 4.38e-10 ***
## doseHI      15.4950     1.2104  12.802 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.828 on 56 degrees of freedom
## Multiple R-squared:  0.7623, Adjusted R-squared:  0.7496
## F-statistic: 59.88 on 3 and 56 DF,  p-value: < 2.2e-16
```

- The model has 4 parameters.
- The  $F$  test at the end compares with the (null) model with only one overall mean parameter.



## 4.4 Testing effect of supp

- Alternative model without effect of supp:

```
doseEff <- lm(len ~ dose, data = ToothGrowth)
```

- We can compare  $R^2$  to see if `doseEff` (Model 1) is sufficient to explain the data compared to `MainEff` (Model 2). This is done by converting to  $F$ -statistic:

$$F_{obs} = \frac{(R_2^2 - R_1^2)/(df_1 - df_2)}{(1 - R_2^2)/df_2} = \frac{(SSE_1 - SSE_2)/(df_1 - df_2)}{(SSE_2)/df_2}.$$

- $SSE_1 - SSE_2$ : increase in error sum of square when using Model 1 instead of Model 2
- In **R** the calculations are done using `anova`:

```
anova(doseEff, MainEff)
```

```
## Analysis of Variance Table
##
## Model 1: len ~ dose
## Model 2: len ~ supp + dose
##   Res.Df    RSS Df Sum of Sq    F    Pr(>F)
## 1      57 1025.78
## 2      56  820.43  1    205.35 14.017 0.0004293 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- $p$ -value is 0.004 hence we reject that `supp` does not have an effect. Thus we prefer Model 2.

## 4.5 Testing effect of dose

- Alternative model without effect of dose:

```
suppEff <- lm(len ~ supp, data = ToothGrowth)
anova(suppEff, MainEff)
```

```
## Analysis of Variance Table
##
## Model 1: len ~ supp
## Model 2: len ~ supp + dose
##   Res.Df    RSS Df Sum of Sq    F    Pr(>F)
## 1      58 3246.9
## 2      56  820.4  2    2426.4 82.811 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

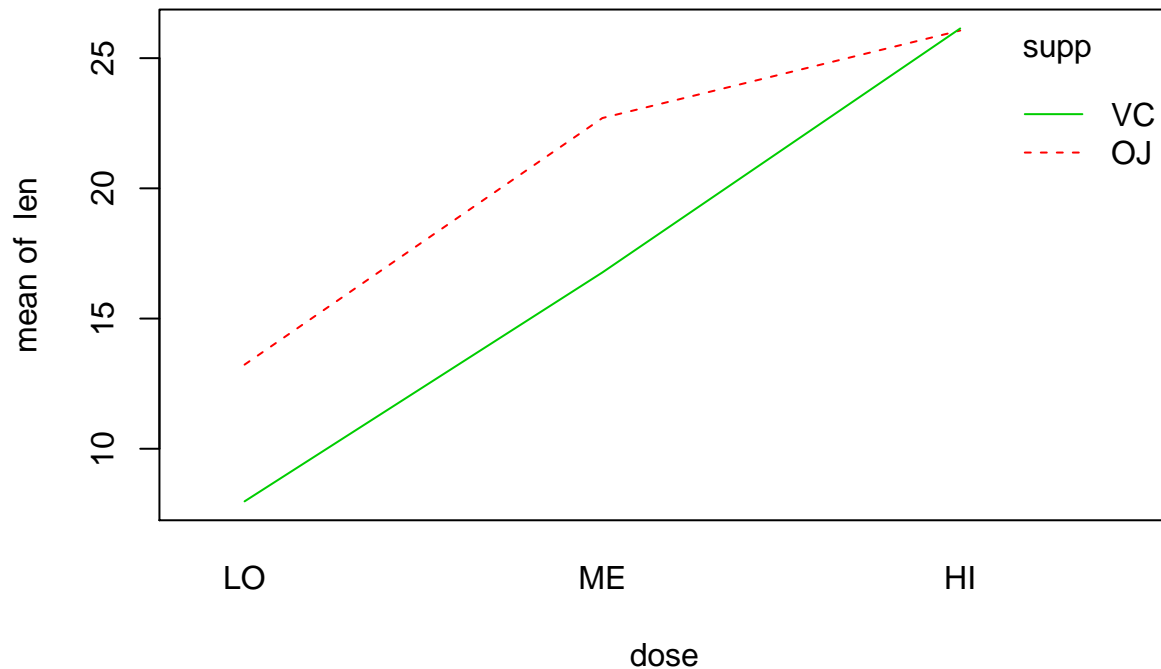
- $p$ -value is  $\approx 0$  hence we reject that `dose` does not have an effect. Thus we prefer Model 2.

## 5 Interaction

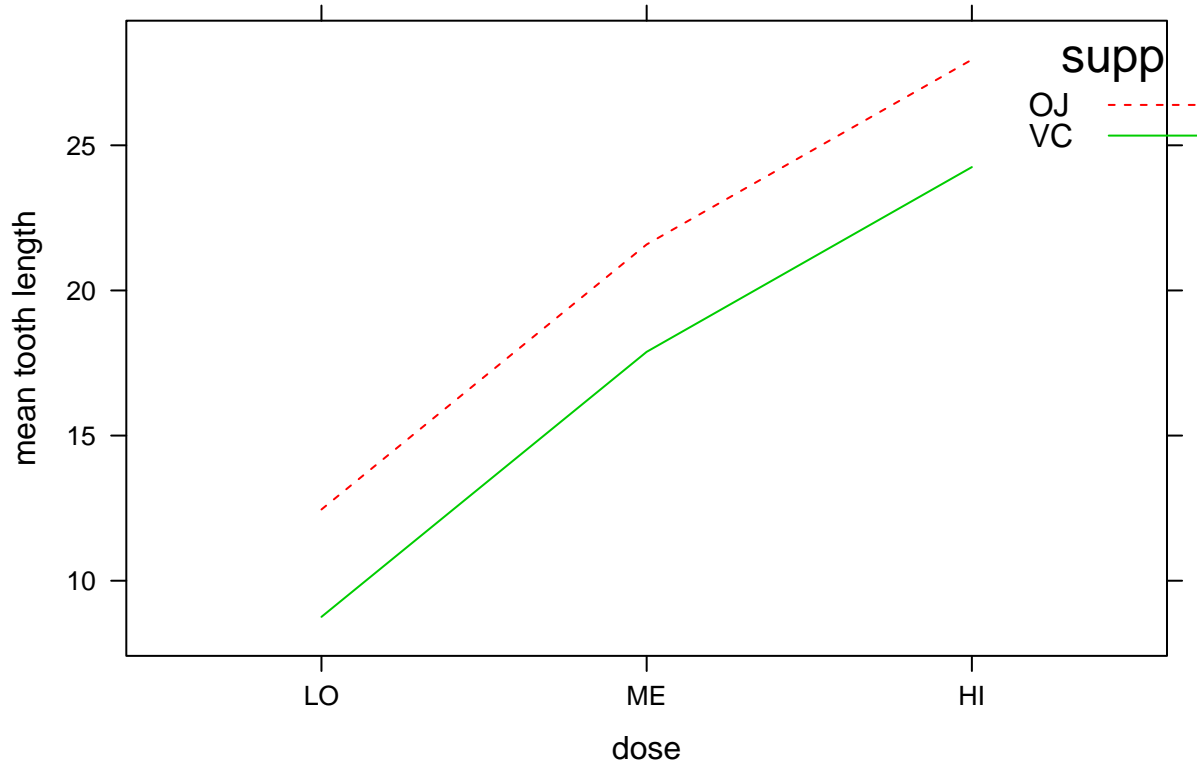
### 5.1 Example

- We will extend the model by introducing an interaction between `supp` and `dose`.
- Interaction plot:

```
with(ToothGrowth, interaction.plot(dose, supp, len, col = 2:3))
```



- For each of the supplement types we plot the average tooth length as a function of dosage.
- If the main effects model is correct then the difference between supplements is the same for all levels of dosage, i.e. the curves should be parallel - except for noise.
- This does not seem to be the case.
- This is how the plot *should* look *if* the main effects model (no interaction) is correct:



- Parallel lines mean that effect of supplement does not depend on dose !

## 5.2 Dummy coding

- The extended model can be formulated as

$$\text{length} = \mu + \beta_1 s_C + \beta_2 d_M + \beta_3 d_H + \beta_4 s_C d_M + \beta_5 s_C d_H + \text{error}$$

- Interpretation:

- $\mu$  is the expected tooth length for **supp OJ** and **dose LO** ( $s_C = d_M = d_H = 0$ ).
- $\beta_1$  is the effect of changing from **supp OJ** to **VC**, **dose** is **LO** ( $s_C = 1, d_M = d_H = 0$ ).
- $\beta_2$  is the effect of increasing **dose** from **LO** to **ME**, when **supp** is **OJ** ( $s_C = 0, d_M = 1$ ).
- $\beta_3$  is the effect of increasing **dose** from **LO** to **HI**, when **supp** is **OJ** ( $s_C = 0, d_H = 1$ ).
- $\beta_4$  is an additional effect of both changing from **supp OJ** to **VC** and increasing **dose** from **LO** to **ME** ( $s_C = 1, d_M = 1$ )
- $\beta_5$  is an additional effect of both changing from **supp OJ** to **VC** and increasing **dose** from **LO** to **HI** ( $s_C = 1, d_H = 1$ )

- As a two-way table:

	<i>LO</i>	<i>ME</i>	<i>HI</i>
<i>OJ</i>	$\mu$	$\mu + \beta_2$	$\mu + \beta_3$
<i>VC</i>	$\mu + \beta_1$	$\mu + \beta_1 + \beta_2 + \beta_4$	$\mu + \beta_1 + \beta_3 + \beta_5$

- Further examples:

- effect of changing from **supp OJ** to **VC** if **dose** is **LO** is  $\mu + \beta_1 - \mu = \beta_1$
- effect of changing from **supp OJ** to **VC** if **dose** is **ME** is  $\mu + \beta_1 + \beta_2 + \beta_4 - \mu - \beta_2 = \beta_1 + \beta_4$
- effect of changing from **supp OJ** to **VC** if **dose** is **HI** is  $\mu + \beta_1 + \beta_3 + \beta_5 - \mu - \beta_3 = \beta_1 + \beta_5$
- if  $\beta_4 = 0$  and  $\beta_5 = 0$  the effect of changing from **OJ** to **VC** does not depend on **dose**

### 5.3 Example

- We fit the interaction model by changing plus to multiply in the model expression from before:

```
Interaction <- lm(len ~ supp*dose, data = ToothGrowth)
```

- Now we can think of an experiment with 6 groups corresponding to each combination of the predictors.
- Is added interaction significant ? - we compare main effects model and more complex interaction model using anova:

```
anova(MainEff, Interaction)
```

```
## Analysis of Variance Table
##
## Model 1: len ~ supp + dose
## Model 2: len ~ supp * dose
##   Res.Df    RSS Df Sum of Sq    F Pr(>F)
## 1      56 820.43
## 2      54 712.11  2    108.32 4.107 0.02186 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- With a p-value of 2.1860269% there is a significant interaction `supp:dose`, i.e. the lack of parallel curves in the interaction plot is significant.

```
summary(Interaction)
```

```
##
## Call:
## lm(formula = len ~ supp * dose, data = ToothGrowth)
##
## Residuals:
##   Min     1Q   Median     3Q      Max
## -8.20  -2.72  -0.27   2.65   8.27
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)    13.230     1.148   11.521 3.60e-16 ***
## suppVC         -5.250     1.624   -3.233 0.00209 **
## doseME          9.470     1.624    5.831 3.18e-07 ***
## doseHI         12.830     1.624    7.900 1.43e-10 ***
## suppVC:doseME  -0.680     2.297   -0.296 0.76831
## suppVC:doseHI  5.330     2.297    2.321 0.02411 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.631 on 54 degrees of freedom
## Multiple R-squared:  0.7937, Adjusted R-squared:  0.7746
## F-statistic: 41.56 on 5 and 54 DF,  p-value: < 2.2e-16
```

- Note the negative effect of changing from OJ to VC when dose is low is cancelled by the positive interaction parameter  $\beta_5 = \text{suppVC:doseHI}$  meaning almost no difference between OJ and VC when dose is high (compare with interaction plot)

## 5.4 Hierarchical principle

- In presence of interaction effect it does not make sense to make tests for absence of main effects ! Indeed each factor has an effect that just happens to vary depending on the other factor
- Hence start by investigating whether there is an interaction effect
- If yes: no further tests !
- If no: you may test main effects if relevant for your study