

Chi-square and ordinal tests

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1 Contingency tables

1.1 A contingency table

- The dataset `popularKids`, we study the **association** between the **factors** `Goals` and `Urban.Rural`:
 - `Urban.Rural`: The students were selected from urban, suburban, and rural schools.
 - `Goals`: The students indicated whether good grades, athletic ability, or popularity was most important to them.
 - In total 478 students from grades 4-6.
- Based on a sample we make a cross tabulation of the factors and we get a so-called **contingency table** (krydstabel).

```
popKids <- read.delim("https://asta.math.aau.dk/datasets?file=PopularKids.txt")
library(mosaic)
tab <- tally(~Urban.Rural + Goals, data = popKids, margins = TRUE)
tab
```

```
##           Goals
## Urban.Rural Grades Popular Sports Total
##   Rural         57      50     42   149
##   Suburban      87      42     22   151
##   Urban        103      49     26   178
##   Total        247     141     90   478
```

1.2 A conditional distribution

- Another representation of data is the probability distribution of `Goals` for each level of `Urban.Rural`, i.e. the sum in each row of the table is 1 (up to rounding):

```
##           Goals
## Urban.Rural Grades Popular Sports  Sum
##   Rural      0.383  0.336  0.282 1.000
##   Suburban   0.576  0.278  0.146 1.000
##   Urban      0.579  0.275  0.146 1.000
##   Total      0.517  0.295  0.188 1.000
```

- Here we will talk about the **conditional distribution** of `Goals` given `Urban.Rural`.
- An important question could be:
 - Are the goals of the kids different when they come from urban, suburban or rural areas? I.e. are the rows in the table significantly different?
- There is (almost) no difference between urban and suburban, but it looks like rural is different.

1.3 Independence

- Recall, that two factors are **independent**, when there is no difference between the population's distributions of one factor given the levels of the other factor.
- Otherwise the factors are said to be **dependent**.
- If we e.g. have the following conditional **population distributions** of Goals given Urban.Rural:

```
##           Goals
## Urban.Rural Grades Popular Sports
##   Rural      0.5     0.3    0.2
##   Suburban  0.5     0.3    0.2
##   Urban     0.5     0.3    0.2
```

- Then the factors Goals and Urban.Rural are independent.
- We take a sample and “measure” the factors F_1 and F_2 . E.g. Goals and Urban.Rural for a random child.
- The hypothesis of interest today is:

$$H_0 : F_1 \text{ and } F_2 \text{ are independent, } H_a : F_1 \text{ and } F_2 \text{ are dependent.}$$

1.4 The Chi-squared test for independence

- Our best guess of the distribution of Goals is the relative frequencies in the sample:

```
tab <- tally(~Urban.Rural + Goals, data = popKids)
n <- margin.table(tab)
pctGoals <- round(margin.table(tab, 2) / n, 3)
pctGoals
```

```
## Goals
## Grades Popular Sports
## 0.517 0.295 0.188
```

- If we assume independence, then this is also a guess of the conditional distributions of Goals given Urban.Rural.
- The corresponding expected counts in the sample are then:

```
##           Goals
## Urban.Rural Grades      Popular      Sports      Sum
##   Rural      77.0 (0.517)  44.0 (0.295)  28.1 (0.188) 149.0 (1.000)
##   Suburban  78.0 (0.517)  44.5 (0.295)  28.4 (0.188) 151.0 (1.000)
##   Urban     92.0 (0.517)  52.5 (0.295)  33.5 (0.188) 178.0 (1.000)
##   Sum       247.0 (0.517) 141.0 (0.295)  90.0 (0.188) 478.0 (1.000)
```

1.5 Calculation of expected table

```
pctexptab
```

```
##           Goals
## Urban.Rural Grades      Popular      Sports      Sum
##   Rural    77.0 (0.517)  44.0 (0.295)  28.1 (0.188) 149.0 (1.000)
##   Suburban 78.0 (0.517)  44.5 (0.295)  28.4 (0.188) 151.0 (1.000)
##   Urban    92.0 (0.517)  52.5 (0.295)  33.5 (0.188) 178.0 (1.000)
##   Sum      247.0 (0.517) 141.0 (0.295)  90.0 (0.188) 478.0 (1.000)
```

- We note that
 - The relative frequency for a given column is **column total** divided by **table total**. For example **Grades**, which is $\frac{247}{478} = 0.517$.
 - The expected value in a given cell in the table is then the cell's relative column frequency multiplied by the cell's **row total**. For example **Rural** and **Grades**: $149 \times 0.517 = 77.0$.
- This can be summarized to:
 - The expected value in a cell is the product of the cell's **row total** and **column total** divided by the **table total**

1.6 Chi-squared (χ^2) test statistic

- We have an **observed table**:

```
tab
```

```
##           Goals
## Urban.Rural Grades Popular Sports
##   Rural      57      50      42
##   Suburban   87      42      22
##   Urban     103      49      26
```

- And an **expected table**, if H_0 is true:

```
##           Goals
## Urban.Rural Grades Popular Sports Sum
##   Rural    77.0  44.0  28.1 149.0
##   Suburban 78.0  44.5  28.4 151.0
##   Urban    92.0  52.5  33.5 178.0
##   Sum      247.0 141.0  90.0 478.0
```

- If these tables are “far from each other”, then we reject H_0 . We want to measure the distance via the Chi-squared test statistic:
 - $X^2 = \sum \frac{(f_o - f_e)^2}{f_e}$: Sum over all cells in the table
 - f_o is the frequency in a cell in the observed table
 - f_e is the corresponding frequency in the expected table.
- We have:

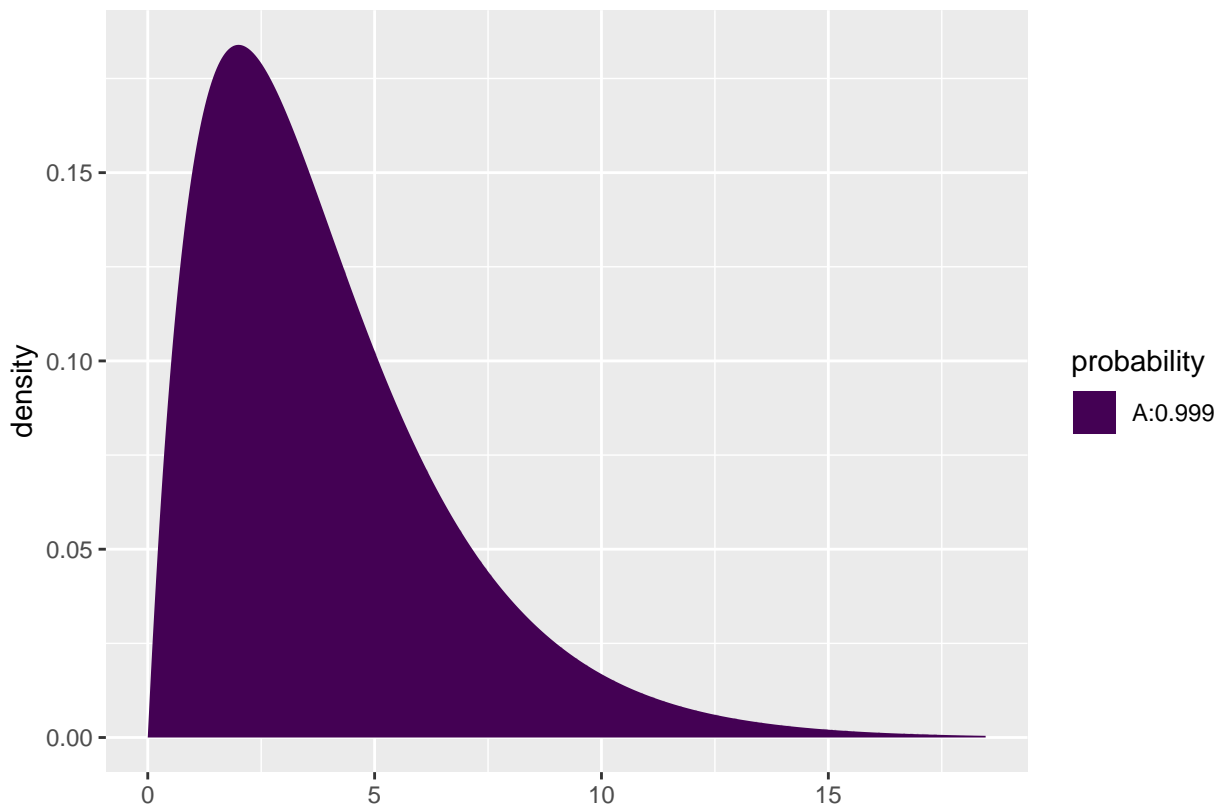
$$X_{obs}^2 = \frac{(57 - 77)^2}{77} + \dots + \frac{(26 - 33.5)^2}{33.5} = 18.8$$

- Is this a large distance??

1.7 χ^2 -test template.

- We want to test the hypothesis H_0 of independence in a table with r rows and c columns:
 - We take a sample and calculate X_{obs}^2 - the observed value of the test statistic.
 - p-value: Assume H_0 is true. What is then the chance of obtaining a larger X^2 than X_{obs}^2 , if we repeat the experiment?
- This can be approximated by the χ^2 -**distribution** with $df = (r - 1)(c - 1)$ degrees of freedom.
- For Goals and Urban.Rural we have $r = c = 3$, i.e. $df = 4$ and $X_{obs}^2 = 18.8$, so the p-value is:

```
1 - pdist("chisq", 18.8, df = 4)
```



```
## [1] 0.0008603303
```

- There is clearly a significant association between Goals and Urban.Rural.

1.8 The function `chisq.test`

- All of the above calculations can be obtained by the function `chisq.test`.

```
tab <- tally(~ Urban.Rural + Goals, data = popKids)
testStat <- chisq.test(tab, correct = FALSE)
testStat
```

```
##
## Pearson's Chi-squared test
##
## data:  tab
## X-squared = 18.828, df = 4, p-value = 0.0008497
```

```
testStat$expected
```

```
##           Goals
## Urban.Rural  Grades  Popular  Sports
##   Rural    76.99372 43.95188 28.05439
##   Suburban 78.02720 44.54184 28.43096
##   Urban   91.97908 52.50628 33.51464
```

-
- The frequency data can also be put directly into a matrix.

```
data <- c(57, 87, 103, 50, 42, 49, 42, 22, 26)
tab <- matrix(data, nrow = 3, ncol = 3)
row.names(tab) <- c("Rural", "Suburban", "Urban")
colnames(tab) <- c("Grades", "Popular", "Sports")
tab
```

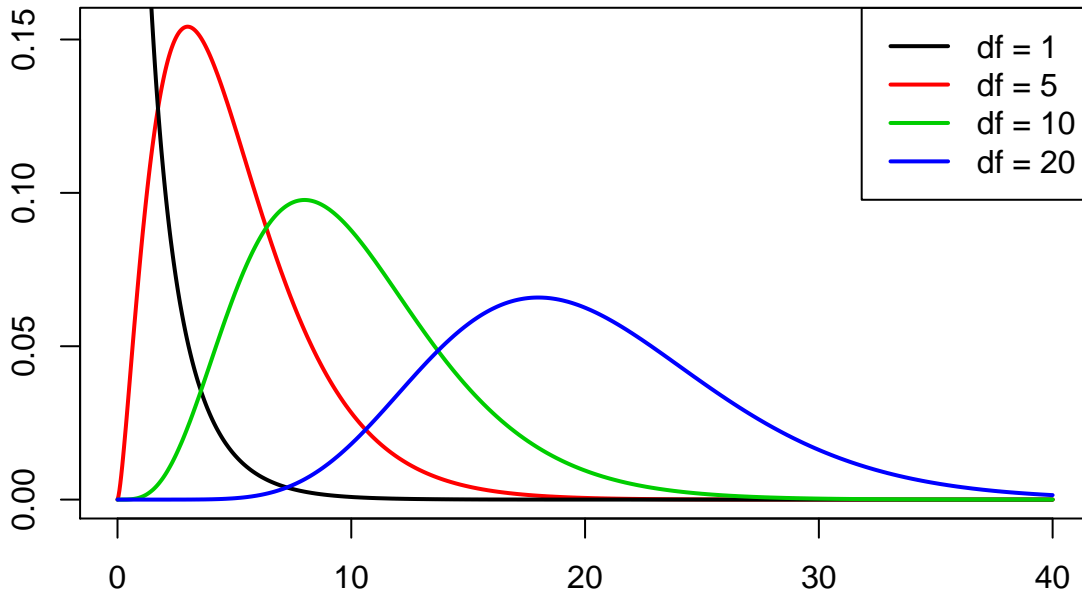
```
##           Grades Popular Sports
## Rural         57      50     42
## Suburban      87      42     22
## Urban        103      49     26
```

```
chisq.test(tab)
```

```
##
## Pearson's Chi-squared test
##
## data:  tab
## X-squared = 18.828, df = 4, p-value = 0.0008497
```

1.9 The χ^2 -distribution

- The χ^2 -distribution with df degrees of freedom:
 - Is never negative. And $X^2 = 0$ only happens if $f_e = f_o$.
 - Has mean $\mu = df$
 - Has standard deviation $\sigma = \sqrt{2df}$
 - Is skewed to the right, but approaches a normal distribution when df grows.



1.10 Summary

- For the the Chi-squared statistic, χ^2 , to be appropriate we require that the expected values have to be $f_e \geq 5$.
- Now we can summarize the ingredients in the Chi-squared test for independence.

TABLE 8.5: The Five Parts of the Chi-Squared Test of Independence

1. Assumptions: Two categorical variables, random sampling, $f_e \geq 5$ in all cells
2. Hypotheses: H_0 : Statistical independence of variables H_a : Statistical dependence of variables
3. Test statistic: $\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e}$, where $f_e = \frac{(\text{Row total})(\text{Column total})}{\text{Total sample size}}$
4. P -value: $P =$ right-tail probability above observed χ^2 value, for chi-squared distribution with $df = (r - 1)(c - 1)$
5. Conclusion: Report P -value If decision needed, reject H_0 at α -level if $P \leq \alpha$

1.11 Residual analysis

- If we reject the hypothesis of independence it can be of interest to identify the significant deviations.
- In a given cell in the table, $f_o - f_e$ is the deviation between data and the expected values under the null hypothesis.
- We assume that $f_e \geq 5$.
- If H_0 is true, then the standard error of $f_o - f_e$ is given by

$$se = \sqrt{f_e(1 - \text{row proportion})(1 - \text{column proportion})}$$

- The corresponding z -score

$$z = \frac{f_o - f_e}{se}$$

should in 95% of the cells be between ± 2 . Values above 3 or below -3 should not appear.

- In `popKids` table cell `Rural` and `Grade` we got $f_e = 77.0$ and $f_o = 57$. Here **column proportion** = 0.517 and **row proportion** = $149/478 = 0.312$.
- We can then calculate

$$z = \frac{57 - 77}{\sqrt{77(1 - 0.517)(1 - 0.312)}} = -3.95$$

- Compared to the null hypothesis there are way too few rural kids who find grades important.
- In summary: The standardized residuals allow for cell-by-cell (f_e vs f_o) comparison.

1.12 Residual analysis in R

- In R we can extract the standardized residuals from the output of `chisq.test`:

```
tab <- tally(~ Urban.Rural + Goals, data = popKids)
testStat <- chisq.test(tab, correct = FALSE)
testStat$stdres
```

```
##           Goals
## Urban.Rural  Grades  Popular  Sports
## Rural      -3.9508449  1.3096235  3.5225004
## Suburban    1.7666608 -0.5484075 -1.6185210
## Urban       2.0865780 -0.7274327 -1.8186224
```

1.13 Cramér's V

- To measure the strength of the association, the Swedish mathematician Harald Cramér developed a measure which is estimated by

$$V = \sqrt{\frac{X^2}{n \cdot \min(r - 1, c - 1)}}$$

where r and c are the number of columns and rows in the contingency table and n is the sample size.

- Property:
 - Cramér's V lies between 0(no association) and 1(complete association)
- In the situation with the factors `Goals` and `Urban.Rural` from the dataset `popularKids` we get

$$V = \sqrt{\frac{X^2}{n \cdot \min(r - 1, c - 1)}} = \sqrt{\frac{18.8}{479 \cdot \min(3 - 1, 3 - 1)}} = 0.14,$$

which indicates a weak (but significant) association.

- The function `CramerV` in the package `DescTools` gives you the value and a confidence interval


```
library(DescTools)
```

```
##  
## Attaching package: 'DescTools'  
  
## The following object is masked from 'package:mosaic':  
##  
## MAD
```

```
CramerV(tab, conf = 0.95, type = "perc")
```

```
## Cramer V lwr.ci upr.ci  
## 0.14033592 0.06014641 0.19419139
```

2 Ordinal variables

2.1 Association between ordinal variables

- For a random sample of black males the General Social Survey in 1996 asked two questions:
 - Q1: What is your yearly income (`income`)?
 - Q2: How satisfied are you with your job (`satisfaction`)?
- Both measurements are on an ordinal scale.

	VeryD	LittleD	ModerateS	VeryS
< 15k	1	3	10	6
15-25k	2	3	10	7
25-40k	1	6	14	12
> 40k	0	1	9	11

- We might do a chi-square test to see whether Q1 and Q2 are associated, but the test does not exploit the ordinality.
- We shall consider a test that incorporates ordinality.

2.2 Gamma coefficient

- Consider a pair of respondents, where **respondent 1** is below **respondent 2** in relation to Q1.
 - If **respondent 1** is also below **respondent 2** in relation to Q2 then the pair is *concordant*.
 - If **respondent 1** is above **respondent 2** in relation to Q2 then the pair is *disconcordant*.
- Let:

C = the number of concordant pairs in our sample.

D = the number of disconcordant pairs in our sample.

- We define the estimated *gamma coefficient*

$$\hat{\gamma} = \frac{C - D}{C + D} = \underbrace{\frac{C}{C + D}}_{\text{concordant prop.}} - \underbrace{\frac{D}{C + D}}_{\text{discordant prop.}}$$

2.3 Gamma coefficient

- Properties:
 - Gamma lies between -1 og 1
 - The sign tells whether the association is positive or negative
 - Large absolute values correspond to strong association
- The standard error $se(\hat{\gamma})$ on $\hat{\gamma}$ is complicated to calculate, so we leave that to software.
- We can now determine a 95% confidence interval:

$$\hat{\gamma} \pm 1.96se(\hat{\gamma})$$

and if zero is contained in the interval, then there is no significant association, when we perform a test with a 5% significance level.

2.4 Example

- First, we need to install the package `vcdExtra`, which has the function `GKgamma` for calculating gamma. It also has the dataset on job satisfaction and income built-in:

```
library(vcdExtra)
JobSat
```

```
##           satisfaction
## income  VeryD LittleD ModerateS VeryS
## < 15k   1         3         10      6
## 15-25k  2         3         10      7
## 25-40k  1         6         14     12
## > 40k   0         1          9     11
```

```
GKgamma(JobSat, level = 0.90)
```

```
## gamma      : 0.221
## std. error  : 0.117
## CI         : 0.028 0.414
```

- A positive association. Marginally significant at the 10% level, but not so at the 5% level.

3 Validation of data

3.1 Goodness of fit test

- You have collected a sample and want to know, whether the sample is representative for people living in Hirtshals.
- E.g. whether the distribution of gender, age, or profession in the sample do not differ significantly from the distribution in Hirtshals.
- Actually, you know how to do that for binary variables like gender, but not if you e.g. have 6 agegroups.

3.2 Example

- As an example we look at k groups, where data from Hjørring kommune tells us the distribution in Hirtshals is given by the vector

$$\pi = (\pi_1, \dots, \pi_k),$$

where π_i is the proportion which belongs to group number i , $i = 1, 2, \dots, k$ in Hirtshals.

- Consider the sample represented by the vector:

$$O = (O_1, \dots, O_k),$$

where O_i is the observed number of individuals in group number i , $i = 1, 2, \dots, k$.

- The total number of individuals:

$$n = \sum_{i=1}^k O_i.$$

- The expected number of individuals in each group, if we have a sample from Hirtshals:

$$E_i = n\pi_i, \quad i = 1, 2, \dots, k$$

3.3 Goodness of fit test

- We will use the following measure to see how far away the observed is from the expected:

$$X^2 = \sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i}$$

- If this is large we reject the hypothesis that the sample has the same distribution as Hirtshals. The reference distribution is the χ^2 with $k - 1$ degrees of freedom.

3.4 Example

- Assume we have four groups and that the true distribution is given by:

```
k <- 4
pi_vector <- c(0.3, 0.2, 0.25, 0.25)
```

- Assume that we have the following sample:

```
O_vector <- c(74, 72, 40, 61)
```

- Expected number of individuals in each group:

```
n <- sum(O_vector)
E_vector <- n * pi_vector
E_vector
```

```
## [1] 74.10 49.40 61.75 61.75
```

- X^2 statistic:

```
Xsq = sum((O_vector - E_vector)^2 / E_vector)
Xsq
```

```
## [1] 18.00945
```

- p -value:

```
p_value <- 1 - pchisq(Xsq, df = k-1)
p_value
```

```
## [1] 0.0004378808
```

3.5 Test in R

```
Xsq_test <- chisq.test(O_vector, p = pi_vector)
Xsq_test
```

```
##
## Chi-squared test for given probabilities
##
## data:  O_vector
## X-squared = 18.009, df = 3, p-value = 0.0004379
```

- As the hypothesis is rejected, we look at the standardized residuals (z -scores):

```
Xsq_test$stdres
```

```
## [1] -0.01388487  3.59500891 -3.19602486 -0.11020775
```

- We conclude that group 1 and 4 is close to true distribution in Hirtshals, but in groups 2 og 3 we have a significant mismatch.