

# ASTA

*The ASTA team*

## Contents

<b>1</b>	<b>Statistical inference: Hypothesis and test</b>	<b>2</b>
1.1	Concept of hypothesis . . . . .	2
1.2	Significance test . . . . .	2
1.3	Null and alternative hypothesis . . . . .	2
1.4	Test statistic . . . . .	2
1.5	$P$ -value . . . . .	3
1.6	Significance level . . . . .	3
1.7	Significance test for mean . . . . .	3
1.8	One-sided $t$ -test for mean . . . . .	4
1.9	Agresti: Overview of $t$ -test . . . . .	5
1.10	Significance test for proportion . . . . .	5
1.11	Agresti: Overview of tests for mean and proportion . . . . .	9
1.12	Response variable and explanatory variable . . . . .	9
1.13	Dependent/independent samples . . . . .	9
1.14	Comparison of two means . . . . .	10
1.15	Comparison of two means: Independent samples . . . . .	10
1.16	Example: Comparing two means (independent samples) . . . . .	10
1.17	Comparison of two means: confidence interval (independent samples) . . . . .	12
1.18	Comparison of two means: paired $t$ -test (dependent samples) . . . . .	12
<b>2</b>	<b>Comparison of two proportions</b>	<b>13</b>
2.1	Comparison of two proportions . . . . .	13
2.2	Comparison of two proportions: Independent samples . . . . .	13
2.3	Approximate test for comparing two proportions (independent samples) . . . . .	14
2.4	Example: Approximate confidence interval and test for comparing proportions . . . . .	14
2.5	Fisher's exact test . . . . .	15
2.6	Agresti: Overview of comparison of two groups . . . . .	16

# 1 Statistical inference: Hypothesis and test

## 1.1 Concept of hypothesis

- A **hypothesis** is a statement about a given population. Usually it is stated as a population parameter having a given value or being in a certain interval.
- Examples:
  - Quality control of products: The hypothesis is that the products e.g. have a certain weight, a given power consumption or a minimal durability.
  - Scientific hypothesis: There is no dependence between a company's age and level of return.

## 1.2 Significance test

- A significance test is used to investigate, whether data is contradicting the hypothesis or not.
- If the hypothesis says that a parameter has a certain value, then the test should tell whether the sample estimate is "far" away from this value.
- For example:
  - Waiting times in a queue. We sample  $n$  customers and count how many that have been waiting more than 5 minutes. The company policy is that at most 10% of the customers should wait more than 5 minutes. In a sample of size  $n = 32$  we observe 4 with waiting time above 5 minutes, i.e. the estimated proportion is  $\hat{\pi} = \frac{4}{32} = 12.5\%$ . Is this "much more" than (i.e. significantly different from) 10%?
  - The blood alcohol level of a student is measured 4 times with the values 0.504, 0.500, 0.512, 0.524, i.e. the estimated mean value is  $\bar{y} = 0.51$ . Is this "much different" than a limit of 0.5?

## 1.3 Null and alternative hypothesis

- **The null hypothesis** - denoted  $H_0$  - usually specifies that a population parameter has some given value. E.g. if  $\mu$  is the mean blood alcohol level we can state the null hypothesis
  - $H_0 : \mu = 0.5$ .
- **The alternative hypothesis** - denoted  $H_a$  - usually specifies that the population parameter is contained in a given set of values different than the null hypothesis. E.g. if  $\mu$  again is the population mean of a blood alcohol level measurement, then
  - the null hypothesis is  $H_0 : \mu = 0.5$
  - the alternative hypothesis is  $H_a : \mu \neq 0.5$ .

## 1.4 Test statistic

- We consider a population parameter  $\mu$  and write the null hypothesis

$$H_0 : \mu = \mu_0,$$

where  $\mu_0$  is a known number, e.g.  $\mu_0 = 0.5$ .

- Based on a sample we have an estimate  $\hat{\mu}$ .
- A **test statistic**  $T$  will typically depend on  $\hat{\mu}$  and  $\mu_0$  (we may write this as  $T(\hat{\mu}, \mu_0)$ ) and measure "how far from  $\mu_0$  is  $\hat{\mu}$ ?"
- Often we use  $T(\hat{\mu}, \mu_0) =$  "the number of standard deviations from  $\hat{\mu}$  to  $\mu_0$ ".
- For example it would be very unlikely to be more than 3 standard deviations from  $\mu_0$ , i.e. in that case  $\mu_0$  is probably not the correct value of the population parameter.

## 1.5 $P$ -value

- We consider
  - $H_0$ : a null hypothesis.
  - $H_a$ : an alternative hypothesis.
  - $T$ : a test statistic, where the value calculated based on the current sample is denoted  $t_{obs}$ .
- To investigate the plausibility of  $H_0$ , we measure the evidence against  $H_0$  by the so-called  $p$ -value:
  - The  $p$ -value is the probability of observing a more extreme value of  $T$  (if we were to repeat the experiment) than  $t_{obs}$  *under the assumption that  $H_0$  is true*.
  - “Extremity” is measured relative to the alternative hypothesis; a value is considered extreme if it is “far from”  $H_0$  and “closer to”  $H_a$ .
  - If the  $p$ -value is small then there is a small probability of observing  $t_{obs}$  if  $H_0$  is true, and thus  $H_0$  is not very probable for our sample and we put more support in  $H_a$ , so:

**The smaller the  $p$ -value, the less we trust  $H_0$ .**

- What is a small  $p$ -value? If it is below 5% we say it is **significant** at the 5% level.

## 1.6 Significance level

- We consider
  - $H_0$ : a null hypothesis.
  - $H_a$ : an alternative hypothesis.
  - $T$ : a test statistic, where the value calculated based on the current sample is denoted  $t_{obs}$  and the corresponding  $p$ -value is  $p_{obs}$ .
- Small values of  $p_{obs}$  are critical for  $H_0$ .
- In practice it can be necessary to decide whether or not we are going to reject  $H_0$ .
- The decision can be made if we previously have decided on a so-called  **$\alpha$ -level**, where
  - $\alpha$  is a given percentage
  - we reject  $H_0$ , if  $p_{obs}$  is less than or equal to  $\alpha$
  - $\alpha$  is called the **significance level** of the test
  - typical choices of  $\alpha$  are 5% or 1%.

## 1.7 Significance test for mean

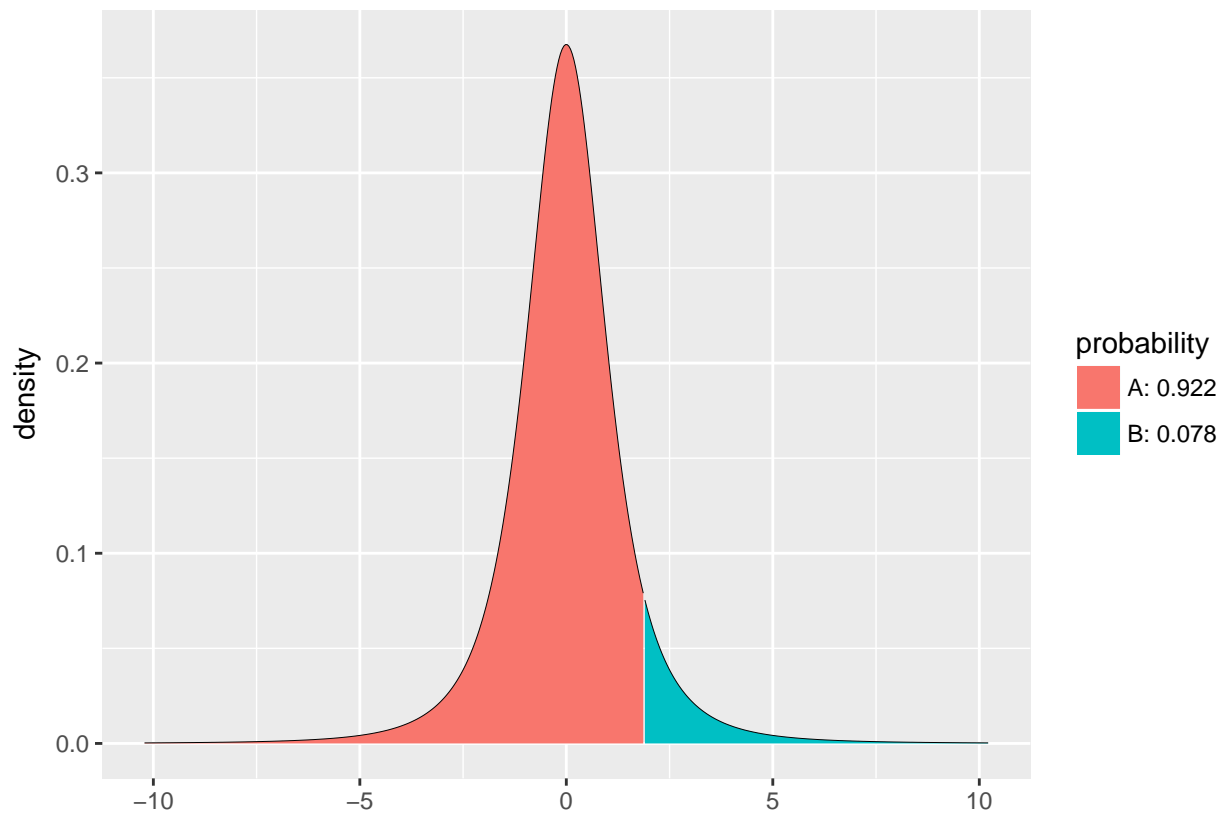
### 1.7.1 Two-sided $t$ -test for mean:

- We assume that data is a sample from  $\text{norm}(\mu, \sigma)$ .
- The estimates of the population parameters are  $\hat{\mu} = \bar{y}$  and  $\hat{\sigma} = s$  based on  $n$  observations.
- Null hypothesis:  $H_0 : \mu = \mu_0$ , where  $\mu_0$  is a known value.
- **Two-sided alternative hypothesis:**  $H_a : \mu \neq \mu_0$ .
- Observed test statistic:  $t_{obs} = \frac{\bar{y} - \mu_0}{se}$ , where  $se = \frac{s}{\sqrt{n}}$ .
- I.e.  $t_{obs}$  measures, how many standard deviations (with  $\pm$  sign) the empirical mean lies away from  $\mu_0$ .
- If  $H_0$  is true, then  $t_{obs}$  is an observation from the  $t$ -distribution with  $df = n - 1$ .
- $P$ -value: Values bigger than  $|t_{obs}|$  or less than  $-|t_{obs}|$  puts more support in  $H_a$  than  $H_0$ .
- The  $p$ -value = 2 x “upper tail probability of  $|t_{obs}|$ ”. The probability is calculated in the  $t$ -distribution with  $df$  degrees of freedom.

### 1.7.2 Example: Two-sided $t$ -test

- Blood alcohol level measurements: 0.504, 0.500, 0.512, 0.524.
- These are assumed to be a sample from a normal distribution.
- We calculate
  - $\bar{y} = 0.51$  and  $s = 0.0106$
  - $se = \frac{s}{\sqrt{n}} = \frac{0.0106}{\sqrt{4}} = 0.0053$ .
  - $H_0 : \mu = 0.5$ , i.e.  $\mu_0 = 0.5$ .
  - $t_{obs} = \frac{\bar{y} - \mu_0}{se} = \frac{0.51 - 0.5}{0.0053} = 1.89$ .
- So we are almost 2 standard deviations from 0.5. Is this extreme in a  $t$ -distribution with 3 degrees of freedom?

```
library(mosaic)
1 - pdist("t", q = 1.89, df = 3)
```



```
## [1] 0.07757725
```

- The  $p$ -value is  $2 \cdot 0.078$ , i.e. more than 15%. On the basis of this we do not reject  $H_0$ .

### 1.8 One-sided $t$ -test for mean

The book also discusses one-sided  $t$ -tests for the mean, but we will not use those in the course.

## 1.9 Agresti: Overview of $t$ -test

**TABLE 6.3:** The Five Parts of Significance Tests for Population Means

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1.	<b>Assumptions</b> Quantitative variable Randomization Normal population (robust, especially for two-sided $H_a$ , large $n$ )
2.	<b>Hypotheses</b> $H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$ (or $H_a: \mu > \mu_0$ or $H_a: \mu < \mu_0$ )
3.	<b>Test statistic</b> $t = \frac{\bar{y} - \mu_0}{se} \text{ where } se = \frac{s}{\sqrt{n}}$
4.	<b>P-value</b> In $t$ curve, use $P$ = Two-tail probability for $H_a: \mu \neq \mu_0$ $P$ = Probability to right of observed $t$ -value for $H_a: \mu > \mu_0$ $P$ = Probability to left of observed $t$ -value for $H_a: \mu < \mu_0$
5.	<b>Conclusion</b> Report $P$ -value. Smaller $P$ provides stronger evidence against $H_0$ and supporting $H_a$ . Can reject $H_0$ if $P \leq \alpha$ -level.

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## 1.10 Significance test for proportion

- Consider a sample of size  $n$ , where we observe whether a given property is present or not.
- The relative frequency of the property in the population is  $\pi$ , which is estimated by  $\hat{\pi}$ .
- Null hypothesis:  $H_0 : \pi = \pi_0$ , where  $\pi_0$  is a known number.
- **Two-sided alternative** hypothesis:  $H_a : \pi \neq \pi_0$ .
- If  $H_0$  is true the standard error for  $\hat{\pi}$  is given by  $se_0 = \sqrt{\frac{\pi_0(1-\pi_0)}{n}}$ .
- Observed test statistic:  $z_{obs} = \frac{\hat{\pi} - \pi_0}{se_0}$
- I.e.  $z_{obs}$  measures, how many standard deviations (with  $\pm$  sign) there is from  $\hat{\pi}$  to  $\pi_0$ .

### 1.10.1 Approximate test

- If both  $n\hat{\pi}$  and  $n(1 - \hat{\pi})$  are larger than 15 we know from previously that  $\hat{\pi}$  follows a normal distribution (approximately), i.e.
  - If  $H_0$  is true, then  $z_{obs}$  is an observation from the standard normal distribution.
- $P$ -value for **two-sided** test: Values greater than  $|z_{obs}|$  or less than  $-|z_{obs}|$  point more towards  $H_a$  than  $H_0$ .

- The  $p$ -value = 2 x “upper tail probability for  $|z_{obs}|$ ”. The probability is calculated in the standard normal distribution.

### 1.10.2 Example: Approximate test

- We consider a study from Florida Poll 2006:
  - In connection with problems financing public service a random sample of 1200 individuals were asked whether they preferred less service or tax increases.
  - 52% preferred tax increases. Is this enough to say that the proportion is significantly different from fifty-fifty?
- Sample with  $n = 1200$  observations and estimated proportion  $\hat{\pi} = 0.52$ .
- Null hypothesis  $H_0 : \pi = 0.5$ .
- Alternative hypothesis  $H_a : \pi \neq 0.5$ .
- Standard error  $se_0 = \sqrt{\frac{\pi_0(1-\pi_0)}{n}} = \sqrt{\frac{0.5 \times 0.5}{1200}} = 0.0144$
- Observed test statistic  $z_{obs} = \frac{\hat{\pi} - \pi_0}{se_0} = \frac{0.52 - 0.5}{0.0144} = 1.39$
- “upper tail probability for 1.39” in the standard normal distribution is 0.0823, i.e. we have a  $p$ -value of  $2 \cdot 0.0823 \approx 16\%$ .
- Conclusion: There is not sufficient evidence to reject  $H_0$ , i.e. we do not reject that the preference in the population is fifty-fifty.
- Note, the above calculations can also be performed automatically in **R** by (a little different results due to rounding errors in the manual calculation):

```
count <- 1200 * 0.52 # number of individuals preferring tax increase
prop.test(x = count, n = 1200, correct = F)
```

```
##
## 1-sample proportions test without continuity correction
##
## data: count out of 1200
## X-squared = 1.92, df = 1, p-value = 0.1659
## alternative hypothesis: true p is not equal to 0.5
## 95 percent confidence interval:
## 0.4917142 0.5481581
## sample estimates:
## p
## 0.52
```

### 1.10.3 Binomial (exact) test

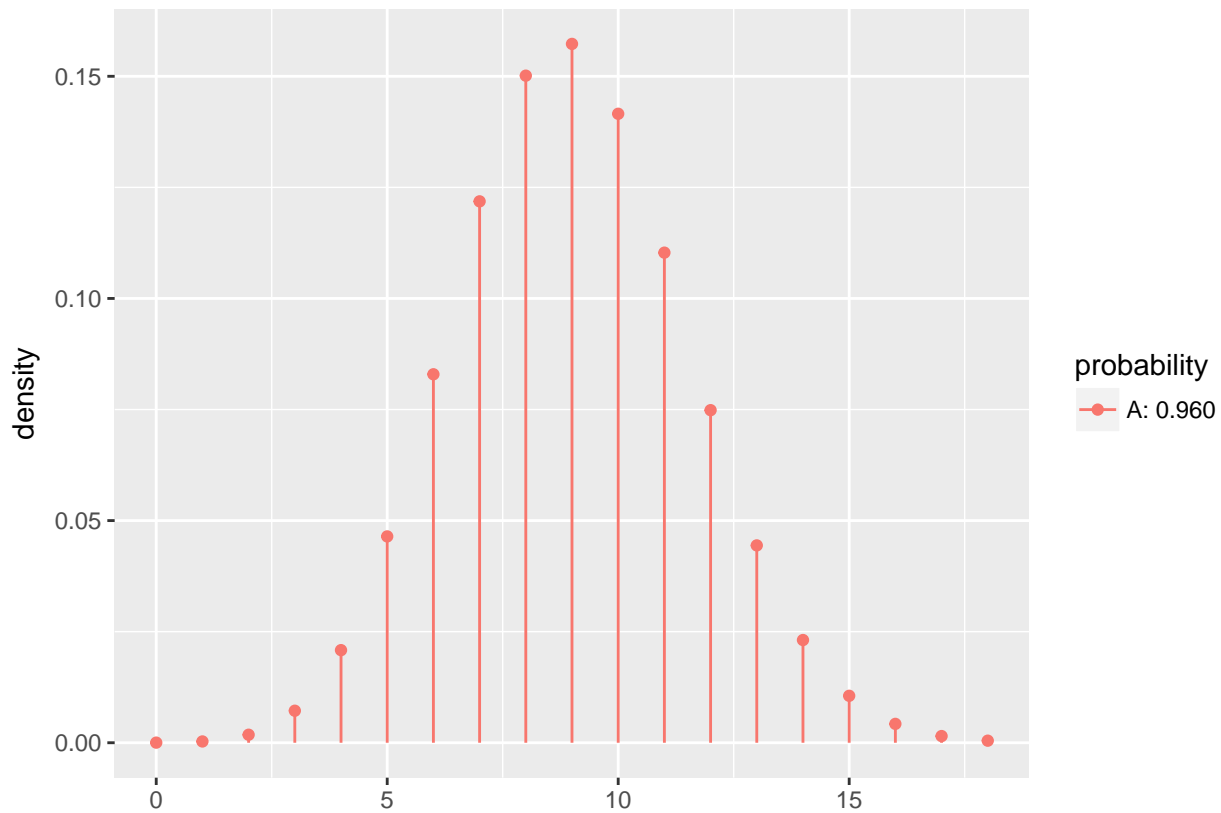
- Consider again a sample of size  $n$ , where we observe whether a given property is present or not.
- The relative frequency of the property in the population is  $\pi$ , which is estimated by  $\hat{\pi}$ .
- Let  $y_+ = n\hat{\pi}$  be the frequency (total count) of the property in the sample.
- It can be shown that  $y_+$  follows the **binomial distribution** with size parameter  $n$  and success probability  $\pi$ . We use  $Bin(n, \pi)$  to denote this distribution.
- Null hypothesis:  $H_0 : \pi = \pi_0$ , where  $\pi_0$  is a known number.
- Alternative hypothesis:  $H_a : \pi \neq \pi_0$ , where  $\pi_0$  is a known number.

- $P$ -value for **two-sided** binomial test:
  - If  $y_+ \geq n\pi_0$ : 2 x “upper tail probability for  $y_+$ ” in the  $Bin(n, \pi_0)$  distribution.
  - If  $y_+ < n\pi_0$ : 2 x “lower tail probability for  $y_+$ ” in the  $Bin(n, \pi_0)$  distribution.

#### 1.10.4 Example: Binomial test

- Experiment with  $n = 30$ , where we have  $y_+ = 14$  successes.
- We want to test  $H_0 : \pi = 0.3$  vs.  $H_a : \pi \neq 0.3$ .
- Since  $y_+ > n\pi_0 = 9$  we use the upper tail probability corresponding to the sum of the height of the red lines to the right of 14 in the graph below. (Notice, the graph continues on the right hand side to  $n = 30$ , but it has been cut off for illustrative purposes.)
- The upper tail probability from 14 and up (i.e. greater than 13) is:

```
lower_tail <- pdist("binom", q = 13, size = 30, prob = 0.3)
```



```
1 - lower_tail
```

```
## [1] 0.04005255
```

- The two-sided  $p$ -value is then  $2 \times 0.04 = 0.08$ .

### 1.10.5 Binomial test in R

- We return to the Chile data, where we again look at the variable `sex`.
- Let us test whether the proportion of females is different from 50 %, i.e., we look at  $H_0 : \pi = 0.5$  and  $H_a : \pi \neq 0.5$ , where  $\pi$  is the unknown population proportion of females.

```
Chile <- read.delim("https://asta.math.aau.dk/datasets?file=Chile.txt")
binom.test( ~ sex, data = Chile, p = 0.5, conf.level = 0.95)
```

```
##
##
##
## data:  Chile$sex  [with success = F]
## number of successes = 1379, number of trials = 2700, p-value =
## 0.2727
## alternative hypothesis: true probability of success is not equal to 0.5
## 95 percent confidence interval:
##  0.4916971 0.5297610
## sample estimates:
## probability of success
##           0.5107407
```

- The  $p$ -value for the binomial exact test is 27%, so there is no significant difference between the proportion of males and females.
- The approximate test has a  $p$ -value of 26%, which can be calculated by the command

```
prop.test( ~ sex, data = Chile, p = 0.5, conf.level = 0.95, correct = FALSE)
```

(note the additional argument `correct = FALSE`).



## 1.11 Agresti: Overview of tests for mean and proportion

TABLE 6.7: Summary of Significance Tests for Means and Proportions

Parameter	Mean	Proportion
1. Assumptions	Random sample, quantitative variable normal population	Random sample, categorical variable null expected counts at least 10
2. Hypotheses	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$ $H_a: \mu > \mu_0$ $H_a: \mu < \mu_0$	$H_0: \pi = \pi_0$ $H_a: \pi \neq \pi_0$ $H_a: \pi > \pi_0$ $H_a: \pi < \pi_0$
3. Test statistic	$t = \frac{\bar{y} - \mu_0}{se}$ with $se = \frac{s}{\sqrt{n}}, df = n - 1$	$z = \frac{\hat{\pi} - \pi_0}{se_0}$ with $se_0 = \sqrt{\pi_0(1 - \pi_0)/n}$
4. P-value	Two-tail probability in sampling distribution for two-sided test ( $H_0: \mu \neq \mu_0$ or $H_a: \pi \neq \pi_0$ ); One-tail probability for one-sided test	
5. Conclusion	Reject $H_0$ if P-value $\leq \alpha$ -level such as 0.05	

## 1.12 Response variable and explanatory variable

- We conduct an experiment, where we at random choose 50 IT-companies and 50 service companies and measure their profit ratio. Is there association between company type (IT/service) and profit ratio?
- In other words we compare samples from 2 different populations. For each company we register:
  - The binary variable `company type`, which is called **the explanatory variable** and divides data in 2 groups.
  - The quantitative variable `profit ratio`, which is called **the response variable**.

## 1.13 Dependent/independent samples

- In the example with profit ratio of 50 IT-companies and 50 service companies we have **independent samples**, since the same company cannot be in both groups.
- Now, think of another type of experiment, where we at random choose 50 IT-companies and measure their profit ratio in both 2009 and 2010. Then we may be interested in whether there is association between year and profit ratio?
- In this example we have **dependent samples**, since the same company is in both groups.
- Dependent samples may also be referred to as paired samples.

## 1.14 Comparison of two means

- We consider the situation, where we have two quantitative samples:
  - Population 1 has mean  $\mu_1$ , which is estimated by  $\hat{\mu}_1 = \bar{y}_1$  based on a sample of size  $n_1$ .
  - Population 2 has mean  $\mu_2$ , which is estimated by  $\hat{\mu}_2 = \bar{y}_2$  based on a sample of size  $n_2$ .
  - We are interested in the difference  $\mu_2 - \mu_1$ , which is estimated by  $d = \bar{y}_2 - \bar{y}_1$ .
  - Assume that we can find the **estimated standard error**  $se_d$  of the difference and that this has degrees of freedom  $df$ .
  - Assume that the samples either are large or come from a normal population.
- Then we can construct a
  - confidence interval for the unknown population difference of means  $\mu_2 - \mu_1$  by

$$(\bar{y}_2 - \bar{y}_1) \pm t_{crit} se_d,$$

where the critical  $t$ -score,  $t_{crit}$ , determines the confidence level.

- significance test:
  - \* for the null hypothesis  $H_0 : \mu_2 - \mu_1 = 0$  and alternative hypothesis  $H_a : \mu_2 - \mu_1 \neq 0$ .
  - \* which uses the test statistic:  $t_{obs} = \frac{(\bar{y}_2 - \bar{y}_1) - 0}{se_d}$ , that has to be evaluated in a  $t$ -distribution with  $df$  degrees of freedom.

## 1.15 Comparison of two means: Independent samples

- In the independent samples situation it can be shown that

$$se_d = \sqrt{se_1^2 + se_2^2},$$

where  $se_1$  and  $se_2$  are estimated standard errors for the sample means in populations 1 and 2, respectively.

- We recall, that for these we have  $se = \frac{s}{\sqrt{n}}$ , i.e.

$$se_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},$$

where  $s_1$  and  $s_2$  are estimated standard deviations for population 1 and 2, respectively.

- **The degrees of freedom**  $df$  for  $se_d$  can be estimated by a complicated formula, which we will not present here.
- For the confidence interval and the significance test we note that:
  - If both  $n_1$  and  $n_2$  are above 30, then we can use the standard normal distribution ( $z$ -score) rather than the  $t$ -distribution ( $t$ -score).
  - If  $n_1$  or  $n_2$  are below 30, then we let **R** calculate the degrees of freedom and  $p$ -value/confidence interval.

## 1.16 Example: Comparing two means (independent samples)

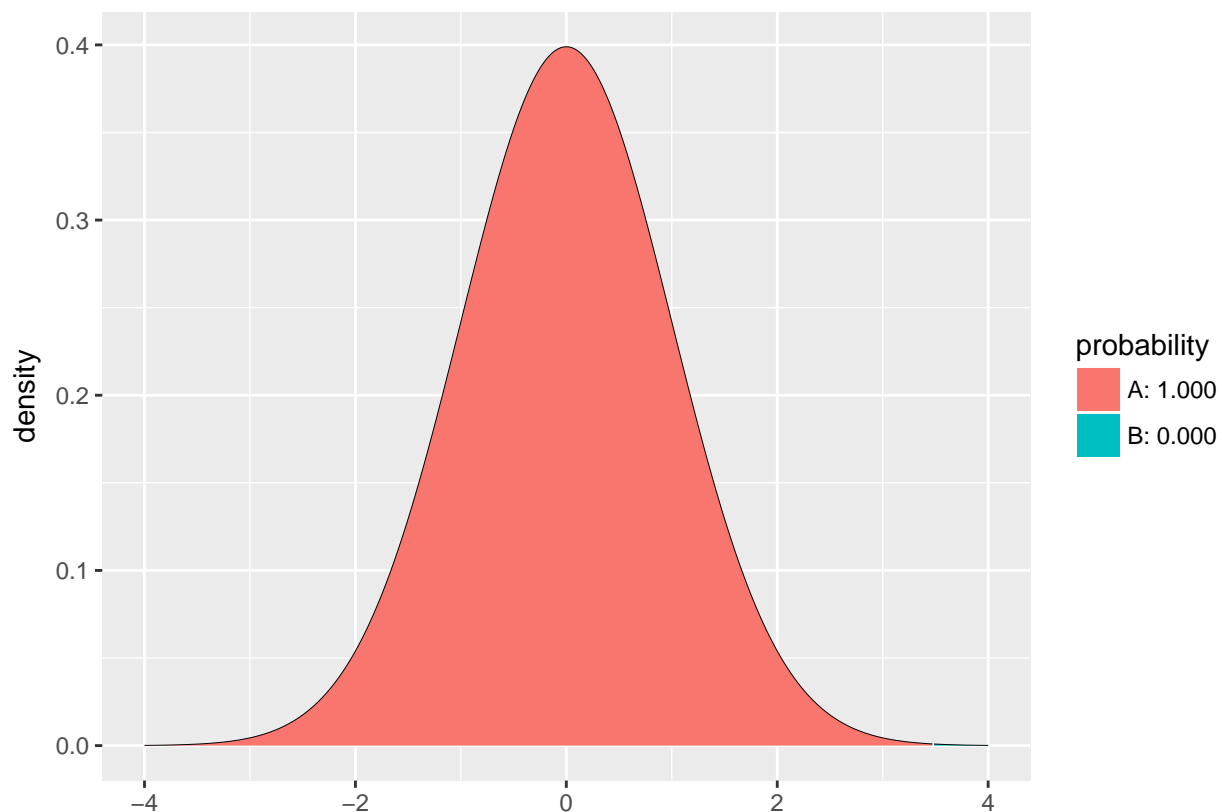
We return to the **Chile** data. We study the association between the variables **sex** and **statusquo** (scale of support for the status-quo). So, we will perform a significance test to test for difference in the mean of **statusquo** for male and females.

```
Chile <- read.delim("https://asta.math.aau.dk/datasets?file=Chile.txt")
library(mosaic)
fv <- favstats(statusquo ~ sex, data = Chile)
fv
```

```
## sex min Q1 median Q3 max mean sd n missing
## 1 F -1.80 -0.975 0.121 1.033 2.02 0.0657 1.003 1368 11
## 2 M -1.74 -1.032 -0.216 0.861 2.05 -0.0684 0.993 1315 6
```

- Difference:  $d = 0.0657 - (-0.0684) = 0.1341$ .
- Estimated standard deviations:  $s_1 = 1.0032$  (females) and  $s_2 = 0.9928$  (males).
- Sample sizes:  $n_1 = 1368$  and  $n_2 = 1315$ .
- Estimated standard error of difference:  $se_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{1.0032^2}{1368} + \frac{0.9928^2}{1315}} = 0.0385$ .
- Observed  $t$ -score for  $H_0 : \mu_1 - \mu_2 = 0$  is:  $t_{obs} = \frac{d-0}{se_d} = \frac{0.1341}{0.0385} = 3.4786$ .
- Since both sample sizes are “pretty large” ( $> 30$ ), we can use the  $z$ -score instead of the  $t$ -score for finding the  $p$ -value (i.e. we use the standard normal distribution):

```
1 - pdist("norm", q = 3.4786, xlim = c(-4, 4))
```



```
## [1] 0.0002520202
```

- Then the  $p$ -value is  $2 \cdot 0.00025 = 0.0005$ , so we reject the null hypothesis.
- We can leave all the calculations to **R** by using `t.test`:

```
t.test(statusquo ~ sex, data = Chile)
```

```
##
## Welch Two Sample t-test
##
## data: statusquo by sex
```

```
## t = 3.4786, df = 2678.7, p-value = 0.0005121
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
## 0.05849179 0.20962982
## sample estimates:
## mean in group F mean in group M
## 0.06570627 -0.06835453
```

- We recognize the  $t$ -score 3.4786 and the  $p$ -value 0.0005. The estimated degrees of freedom  $df = 2679$  is so large that we can not tell the difference between results obtained using  $z$ -score and  $t$ -score.

## 1.17 Comparison of two means: confidence interval (independent samples)

- We have already found all the ingredients to construct a **confidence interval for  $\mu_2 - \mu_1$** :

- $d = \bar{y}_2 - \bar{y}_1$  estimates  $\mu_2 - \mu_1$ .
- $se_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$  estimates the standard error of  $d$ .

- Then:

$$d \pm t_{crit} se_d$$

is a confidence interval for  $\mu_2 - \mu_1$ .

- The critical  $t$ -score,  $t_{crit}$  is chosen corresponding to the wanted confidence level. If  $n_1$  and  $n_2$  both are greater than 30, then  $t_{crit} = 2$  yields a confidence level of approximately 95%.

## 1.18 Comparison of two means: paired $t$ -test (dependent samples)

- Experiment:
  - You choose 10 Netto stores at random, where you measure the average expedition time by the cash registers over some period of time.
  - Now, new cash registers are installed in all 10 stores, and you repeat the experiment.
- It is interesting to investigate whether or not the new cash registers have changed the expedition time.
- So we have 2 samples corresponding to old/new technology. In this case we have **dependent** samples, since we have 2 measurement in each store.
- We use the following strategy for analysis:
  - For each store calculate **the change** in average expedition time when we change from old to new technology.
  - The changes  $d_1, d_2, \dots, d_{10}$  are now considered as **ONE** sample from a population with mean  $\mu$ .
  - Test the hypothesis  $H_0 : \mu = 0$  as usual (using a  $t$ -test for testing the mean as in the previous lecture).

---

### 1.18.1 Netto store example

- Data is organized in a data frame with 2 variables, **before** and **after**, containing the average expedition time before and after installation of the new technology. Instead of doing manual calculations we let **R** perform the significance test (using `t.test` with `paired = TRUE` as our samples are paired/dependent):

```
Netto <- read.delim("https://asta.math.aau.dk/datasets?file=Netto.txt")
head(Netto, n = 3)
```

```
##      before   after
## 1 3.730611 3.440214
## 2 2.623338 2.314733
## 3 3.795295 3.586334
```

```
t.test(Netto$before, Netto$after, paired = TRUE)
```

```
##
## Paired t-test
##
## data: Netto$before and Netto$after
## t = 5.7204, df = 9, p-value = 0.0002868
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  0.1122744 0.2591578
## sample estimates:
## mean of the differences
##                0.1857161
```

- With a  $p$ -value of 0.00029 we reject that the expedition time is the same after installing new technology.

## 2 Comparison of two proportions

### 2.1 Comparison of two proportions

- We consider the situation, where we have two qualitative samples and we investigate whether a given property is present or not:
  - Let the proportion of population 1 which has the property be  $\pi_1$ , which is estimated by  $\hat{\pi}_1$  based on a sample of size  $n_1$ .
  - Let the proportion of population 2 which has the property be  $\pi_2$ , which is estimated by  $\hat{\pi}_2$  based on a sample of size  $n_2$ .
  - We are interested in the difference  $\pi_2 - \pi_1$ , which is estimated by  $d = \hat{\pi}_2 - \hat{\pi}_1$ .
  - Assume that we can find the **estimated standard error**  $se_d$  of the difference.
- Then we can construct
  - an approximate confidence interval for the difference,  $\pi_2 - \pi_1$ .
  - a significance test.

### 2.2 Comparison of two proportions: Independent samples

- In the situation where we have independent samples we know that

$$se_d = \sqrt{se_1^2 + se_2^2},$$

where  $se_1$  and  $se_2$  are the estimated standard errors for the sample proportion in population 1 and 2, respectively.

- We recall, that these are given by  $se = \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$ , i.e.

$$se_d = \sqrt{\frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_2}}.$$

- A (approximate) confidence interval for  $\pi_2 - \pi_1$  is obtained by the usual construction:

$$(\hat{\pi}_2 - \hat{\pi}_1) \pm z_{crit} se_d,$$

where the critical  $z$ -score determines the confidence level.

## 2.3 Approximate test for comparing two proportions (independent samples)

- We consider the null hypothesis  $H_0: \pi_1 = \pi_2$  (equivalently  $H_0: \pi_1 - \pi_2 = 0$ ) and the alternative hypothesis  $H_a: \pi_1 \neq \pi_2$ .
- Assuming  $H_0$  is true, we have a common proportion  $\pi$ , which is estimated by

$$\hat{\pi} = \frac{n_1\hat{\pi}_1 + n_2\hat{\pi}_2}{n_1 + n_2},$$

i.e. we aggregate the populations and calculate the relative frequency of the property (with other words: we estimate the proportion,  $\pi$ , as if the two samples were one).

- Rather than using the estimated standard error of the difference from previous, we use the following that holds under  $H_0$ :

$$se_0 = \sqrt{\hat{\pi}(1-\hat{\pi}) \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}$$

- The observed test statistic/ $z$ -score for  $H_0$  is then:

$$z_{obs} = \frac{(\hat{\pi}_2 - \hat{\pi}_1) - 0}{se_0},$$

which is evaluated in the standard normal distribution.

- The  $p$ -value is calculated in the usual way.

**WARNING:** The approximation is only good, when  $n_1\hat{\pi}$ ,  $n_1(1-\hat{\pi})$ ,  $n_2\hat{\pi}$ ,  $n_2(1-\hat{\pi})$  all are greater than 5.

## 2.4 Example: Approximate confidence interval and test for comparing proportions

We return to the `Chile` dataset. We make a new binary variable indicating whether the person intends to vote no or something else (and we remember to tell `R` that it should think of this as a grouping variable, i.e. a `factor`):

```
Chile$voteNo <- relevel(factor(Chile$vote == "N"), ref = "TRUE")
```

We study the association between the variables `sex` and `voteNo`:

```
tab <- tally(~ sex + voteNo, data = Chile, useNA = "no")
tab
```

```
##   voteNo
## sex TRUE FALSE
##   F  363   946
##   M  526   697
```

This gives us all the ingredients needed in the hypothesis test:

- Estimated proportion of men that vote no:  $\hat{\pi}_1 = \frac{526}{526+697} = 0.430$
- Estimated proportion of women that vote no:  $\hat{\pi}_2 = \frac{363}{363+946} = 0.277$
- Estimated common proportion:  $\hat{\pi} = \frac{1223 \times 0.430 + 1309 \times 0.277}{1309+1223} = \frac{526+363}{1309+1223} = 0.351$ .
- Estimated difference  $d = \hat{\pi}_2 - \hat{\pi}_1 = 0.277 - 0.430 = -0.153$

Further,

- Standard error of difference:  

$$se_d = \sqrt{\frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_2}} = \sqrt{\frac{0.430(1-0.430)}{1223} + \frac{0.277(1-0.277)}{1309}} = 0.0188.$$
- Approximate 95% confidence interval for difference:  $d \pm 1.96se_d = (-0.190, -0.116)$ .
- Standard error of difference when  $H_0 : \pi_1 = \pi_2$  is true:  

$$se_0 = \sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = 0.0190.$$
- The observed test statistic/z-score:  $z_{obs} = \frac{d}{se_0} = -8.06$ . The test for  $H_0$  against  $H_a : \pi_1 \neq \pi_2$  yields a  $p$ -value that is practically zero, i.e. we can reject that the proportions are equal.

### 2.4.1 Automatic calculation in R

```
Chile2 <- subset(Chile, !is.na(voteNo))
prop.test(voteNo ~ sex, data = Chile2, correct = FALSE)
```

```
##
## 2-sample test for equality of proportions without continuity
## correction
##
## data:  tally(voteNo ~ sex)
## X-squared = 64.777, df = 1, p-value = 8.389e-16
## alternative hypothesis: two.sided
## 95 percent confidence interval:
## -0.1896305 -0.1159275
## sample estimates:
##   prop 1   prop 2
## 0.2773109 0.4300899
```

### 2.5 Fisher's exact test

- If  $n_1\hat{\pi}$ ,  $n_1(1-\hat{\pi})$ ,  $n_2\hat{\pi}$ ,  $n_2(1-\hat{\pi})$  are not all greater than 5, then the approximate test cannot be trusted. Instead you can use Fisher's exact test:

```
fisher.test(tab)
```

```
##
## Fisher's Exact Test for Count Data
##
## data: tab
## p-value = 1.04e-15
## alternative hypothesis: true odds ratio is not equal to 1
## 95 percent confidence interval:
## 0.4292768 0.6021525
## sample estimates:
## odds ratio
## 0.5085996
```

- Again the  $p$ -value is seen to be extremely small, so we definitely reject the null hypothesis of equal vote proportions for women and men.

## 2.6 Agresti: Overview of comparison of two groups

TABLE 7.10: Summary of Comparison Methods for Two Groups, for Independent Random Samples

	Type of Response Variable	
	Categorical	Quantitative
<b>Estimation</b>		
1. Parameter	$\pi_2 - \pi_1$	$\mu_2 - \mu_1$
2. Point estimate	$\hat{\pi}_2 - \hat{\pi}_1$	$\bar{y}_2 - \bar{y}_1$
3. Standard error	$se = \sqrt{\frac{\hat{\pi}_1(1-\hat{\pi}_1)}{n_1} + \frac{\hat{\pi}_2(1-\hat{\pi}_2)}{n_2}}$	$se = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$
4. Confidence interval	$(\hat{\pi}_2 - \hat{\pi}_1) \pm z(se)$	$(\bar{y}_2 - \bar{y}_1) \pm t(se)$
<b>Significance testing</b>		
1. Assumptions	Randomization $\geq 10$ observations in each category, for each group	Randomization Normal population dist.'s (robust, especially for large $n$ 's)
2. Hypotheses	$H_0: \pi_1 = \pi_2$ $(\pi_2 - \pi_1 = 0)$ $H_a: \pi_1 \neq \pi_2$	$H_0: \mu_1 = \mu_2$ $(\mu_2 - \mu_1 = 0)$ $H_a: \mu_1 \neq \mu_2$
3. Test statistic	$z = \frac{\hat{\pi}_2 - \hat{\pi}_1}{se_0}$	$t = \frac{\bar{y}_2 - \bar{y}_1}{se}$
4. $P$ -value	Two-tail probability from standard normal or $t$ (Use one tail for one-sided alternative)	