# Rejection sampling

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## Rejection sampling (or von Neumann sampling or acceptance-rejection method)

Rejection sampling is a general method for simulation of a random variable (or random vector) $X$ with a pdf $f \propto f_0$, assuming there is another random variable (or random vector of the same dimension) $Y$ with pdf $g \propto g_0$ so that $f_0(x) \leq M g_0(x)$ for all possible realizations $x$ of $X$, where $M$ is a positive number.

Rejection sampling requires only that we need to know $g_0$ and $M$, and it works as follows.

Repeat generating independently

- realizations $Y = y$ and $U = u$ until $u \leq f_0(y)/[M g_0(y)]$,
- and then return $X = y$ as a simulation from $f$.

The idea is that if it is hard to simulate directly from $f$, it should be simpler to simulate from $g$.

The algorithm works best if $f$ and $g_0$ are approximately proportional.

If $f = c_1 f_0$ and $g = c_2 g_0$, where $c_1$ and $c_2$ are normalizing constants, then the probability of getting acceptance is

$$P(U \leq f(Y)/[M g_0(Y)]) = \cdots = c_2/(c_1 M),$$

so the best choice of $M$ is $M = c_2/c_1$ (whose value may be unknown).

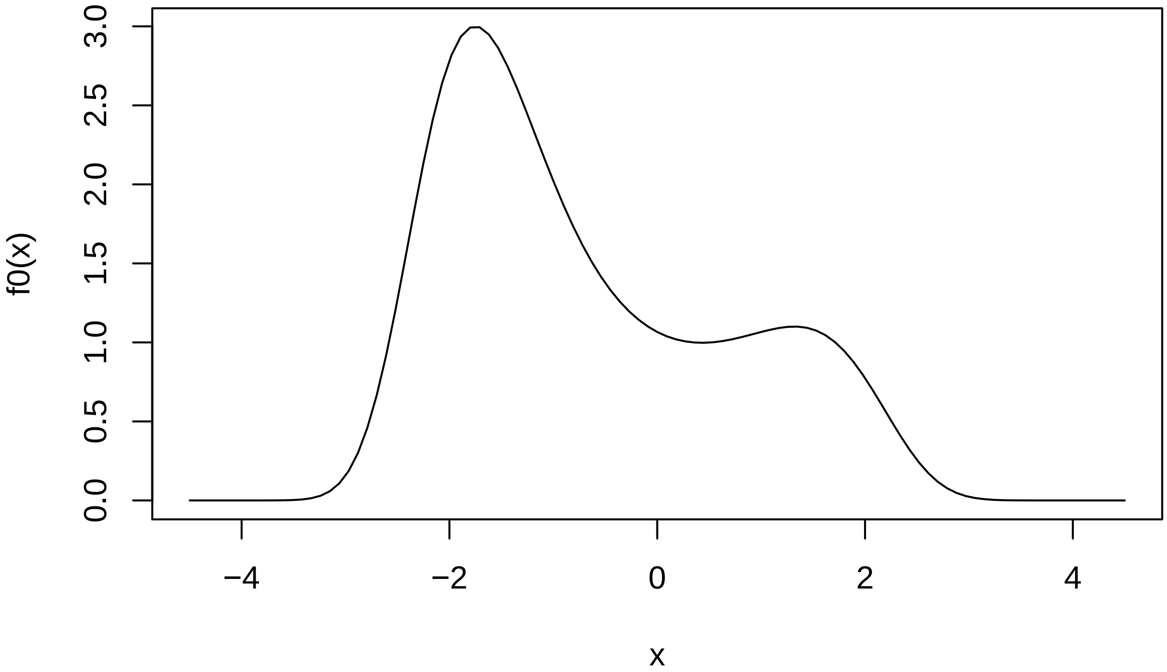
## Target density

Assume that the target density $f$ has support on $I = [-4, 4]$ where it is proportional to

$$f_0(x; a, b) = \exp(a(x-a)^2 - bx^4),$$

where $a, b$ are known parameters (we will use $a = 0.4$ and $b = 0.08$).

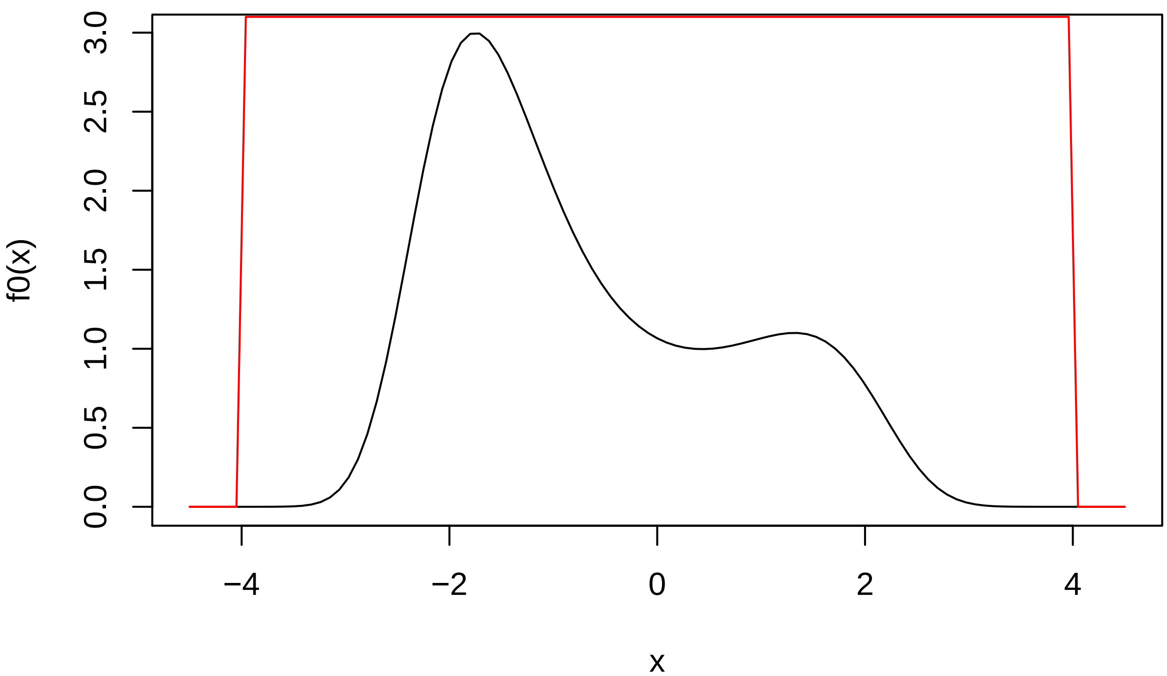
```
f0 <- function(x, a=.4, b=.08){exp(a * (x - a)^2 - b * x^4)}
curve(f0(x), from = -4.5, to = 4.5)
```




### Proposal distribution

Assume we make uniform proposals on $I = [-4, 4]$, i.e. $g(x) = 1/8$ on $I$ and zero outside $I$. Note that $f_0(x) \leq 3.1$ on $I$, so we can use $M = 3.1/(1/8) = 24.8$:

```
curve(f0(x), from = -4.5, to = 4.5)
M <- 24.8
g <- function(x){dunif(x, -4, 4)}
curve(M*g(x), add = TRUE, col = "red")
```



### Running the algorithm

Now we can generate proposals from $g$ (`runif()`) and note whether they are accepted as realizations from the target distribution or not:

```
N <- 10000
y <- runif(N, -4, 4)
p_accept <- f0(y)/(M*g(y))
u <- runif(N, 0, 1)
keep <- u<p_accept
head(round(cbind(y, percent = 100*p_accept, keep), 3))
```

```
##           y percent keep
## [1,]  1.645  33.387    0
## [2,]  0.733  32.952    0
## [3,] -0.043  34.893    0
## [4,]  3.341   0.048    0
## [5,]  1.710  32.336    0
## [6,]  2.911   1.286    0
```

### Results

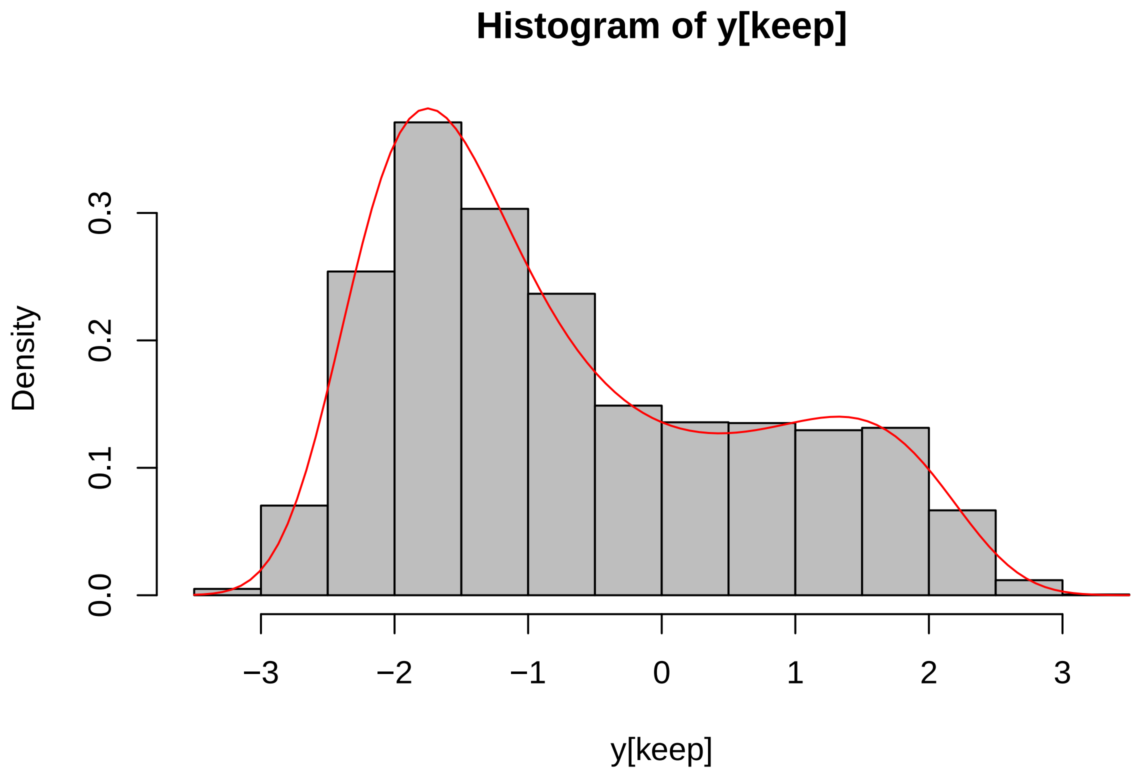
The resulting acceptance rate is

```
mean(keep)
```

```
## [1] 0.3212
```

A histogram of the samples:

```
hist(y[keep], prob = TRUE, col = "gray")
norm_const <- integrate(f0, -4, 4)$value
curve(f0(x)/norm_const, col = "red", add = TRUE)
```


Histogram of y[keep]
Density
0.0
0.1
0.2
0.3
−3
−2
−1
0
1
2
3
y[keep]