Module 3: Binomial model

Ege Rubak and Jesper Møller
Based on material by Søren Højsgaard

Rolling a six

Consider a simple experiment: Roll a die n times; record the number of times six comes up, and denote it y.

Suppose e.g. $n = 10$ and $y = 3$.

We let θ denote the true (but to us unknown) probability of rolling a six (success) and $1 - \theta$ is the probability of a roll less than six (failure):

$$Pr(S) = \theta, \quad Pr(F) = 1 - \theta.$$

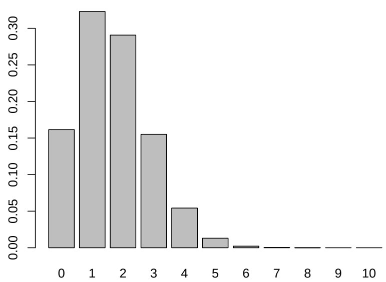
The binomial model

The binomial distribution could be a model for these data: $y = 3$ is a realization of a binomial random variable $Y \sim bin(n, \theta)$.

The density for y is

$$\Pr(Y = y; \theta) = \frac{n!}{y!(n-y)!}\theta^y(1-\theta)^{n-y}.$$

```
yval <- 0:10
barplot(dbinom(yval, size=10, prob=1/6), names.arg=yval)
```



Thus, if we know θ, then we can make all sorts of interesting computations based on the binomial model. E.g. the mean and variance which are given by

$$\mathrm{E}(Y) = n\theta, \quad \mathrm{Var}(Y) = n\theta(1-\theta).$$

Or we can calculate the probability of seeing 0 sixes or the probability of seeing 5 or more sixes when rolling a die 10 times.

For example, if the die is fair and $\theta = 1/6$ we get

```
dbinom(0, size=10, prob=1/6)    # Pr(0 sixes)
```

```
## [1] 0.1615056
```

```r
1-pbinom(4, size=10, prob=1/6) # Pr(5 or more sixes)
```

```
## [1] 0.01546197
```

A moment estimate

In practice θ is unknown and must be estimated from data. Intuition says that θ should be estimated as

$$\hat{\theta} = y/n = 3/10 = 0.3.$$

It is useful to write $\hat{\theta}(y) = y/n$ to emphasize the dependence on data.

For the corresponding random variable $\hat{\theta}(Y) = Y/n$,

$$\mathrm{E}(Y/n) = \theta, \quad \mathrm{Var}(Y/n) = \frac{1}{n^2} n\theta(1-\theta) = \theta(1-\theta)/n.$$

Hence $\hat{\theta}(Y)$ has the correct mean value (unbiased) and the variance of $\hat{\theta}(Y)$ goes to 0 when $n \to \infty$ (consistent).

To calculate the variance, we plug in the estimate and find

$$\mathrm{Var}(Y/n) \approx \frac{y}{n}(1-\frac{y}{n})/n = 0.3 \times 0.7/10 = 0.021$$

I.e. the estimated standard deviation of the estimate (called std. error) is approximately $\sqrt{0.021} = 0.14$.

The likelihood

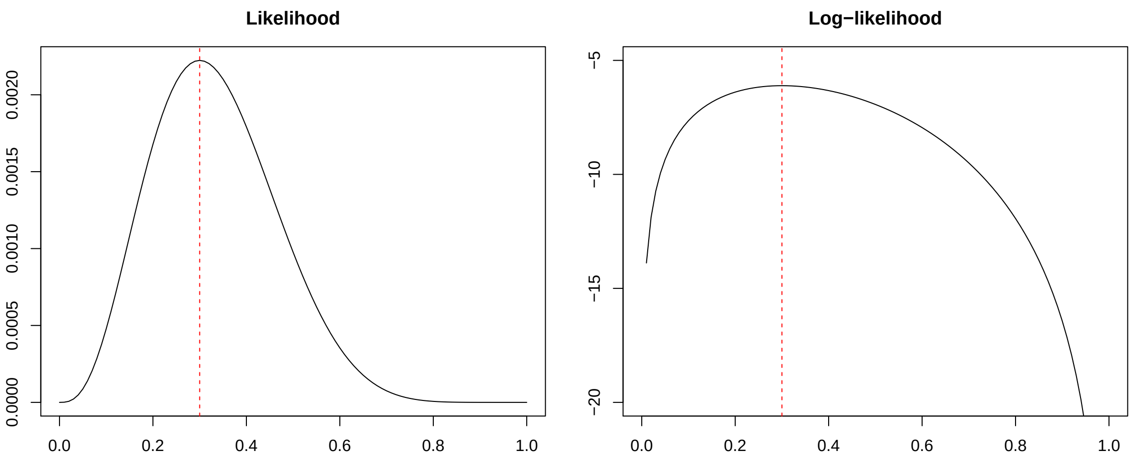
When $y = 3$ is observed, then the binomial density is a function of θ, and it is called the likelihood function:

$$L(\theta) = Pr_\theta(Y = y) = \frac{n!}{y!(n-y)!}\theta^y(1-\theta)^{n-y} \propto \theta^y(1-\theta)^{n-y}.$$

Hence, the log-likelihood is

$$l(\theta) = \log L(\theta) = y \log\theta + (n-y)\log(1-\theta).$$

```r
lik <- function(parm, y, n){parm^y * (1 - parm)^(n - y)}
loglik <- function(parm, y, n){y * log(parm) + (n - y) * log(1 - parm)}
n <- 10; y <- 3
curve(lik(x, y, n), main = "Likelihood")
abline(v = y/n, col = "red", lty = 2)
curve(loglik(x, y, n), main = "Log-likelihood", ylim = c(-20, -5))
abline(v = y/n, col = "red", lty = 2)
```



The frequentist approach

From a frequentist perspective we want to find the "best" estimate of θ given data, and "best" is here the value of θ that maximizes $L()$. The estimate is called the maximum likelihood estimate (MLE).

In practice it is usually easier to maximize $l(\theta) = \log L(\theta)$ instead of $L(\theta)$ because the log turns a product into a sum, and sums are easier to differentiate than products.

The usual approach to maximizing $l(\theta)$ is to first differentiate $l(\theta)$ to obtain

$$S(\theta) = l'(\theta)$$

where $S(\theta)$ is called the **score function**. Next we solve the **score equation** $S(\theta) = 0$ to obtain $\hat{\theta}$.

It is not hard to spot that the maximum of $L()$ (and of $l()$) is at $\theta = y/n = 0.3$, but it is informative to look at the plots for different choices of n and y. This is left as an exercise.

Summary of frequentist approach

- When we have a statistical model (a probability distribution) and data, then we have the likelihood (and the log-likelihood) functions.
- The maximum likelihood estimate (MLE) $\hat{\theta}$ is the value of θ that maximizes the likelihood function.
- The variance of the corresponding estimator is approximately minus the inverse of the second derivative of the log likelihood evaluated at the MLE.

The Bayesian approach

If we take a Bayesian perspective then things change as explained in the main slides for this module:

> The parameter θ is a random quantity on equal footing with Y

and we have to specifiy the prior

$$\pi(\theta).$$

This is our belief about θ before seeing any data, and then given data y we get the posterior from the likelihood via Bayes' rule

$$\pi(\theta|y) = \frac{\pi(y|\theta)\pi(\theta)}{\pi(y)},$$

where $\pi(y|\theta)$ is (proportional to) the likelihood we have specified previously and $\pi(y)$ is the marginal probability for the data y.

When data y is observed then $\pi(y)$ above is just a number, which ensures that $\pi(\theta|y)$ is a density, i.e. that $\int \pi(\theta|y)d\theta = 1$. We do often not calulate $\pi(y)$ directly: We just use that $\pi(\theta|y) \propto \pi(y|\theta)\pi(\theta)$

A discrete prior

Example: Assume (to ease computations) that the only valid choices for θ are now $.1, .3, .5, .7$ and $.9$. Before rolling the die we think the die has been rigged somehow and we take the prior to be

```
theta <- c(.1, .3, .5, .7, .9)
prior <- c(0.10, 0.15, 0.25, 0.30, 0.20)
```

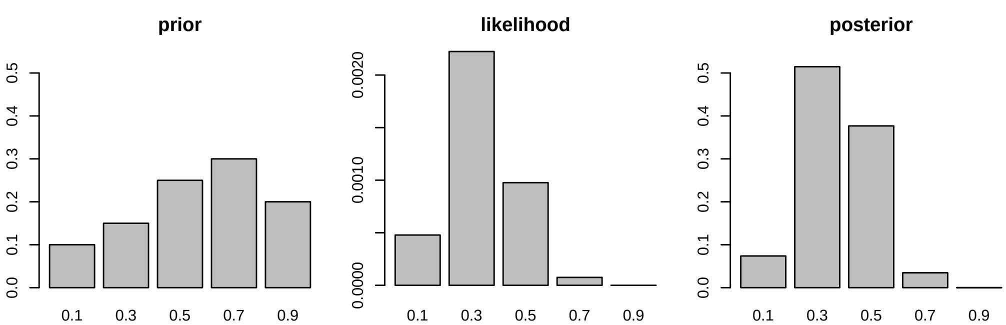
Then we can calculate the likelihood and the posterior

```
n <- 10; y <- 3
likval <- lik(theta, y, n)
posterior <- likval * prior
posterior <- posterior / sum( posterior )
round(100*posterior, 3)
```

```
## [1]  7.381 51.470 37.675  3.473  0.002
```

The plot shows it all:

```
par(mfrow=c(1,3), mar = c(3, 3, 3, 0.5))
barplot(prior, main="prior", names.arg=theta, ylim = c(0, .55))
barplot(likval, main="likelihood", names.arg=theta)
barplot(posterior, main="posterior", names.arg=theta, ylim = c(0, .55))
```



We can compute e.g. the prior and posterior means:

```
sum(theta * prior)
```

```
## [1] 0.57
```

```r
sum(theta * posterior)
```

```
## [1] 0.374492
```

And corresponding variances (uncertanties before and after seeing data).

```r
sum(theta^2 * prior) - sum(theta * prior)^2
```

```
## [1] 0.0611
```

```r
sum(theta^2 * posterior) - sum(theta * posterior)^2
```

```
## [1] 0.01803762
```