Exercises for module 1

Basics of probability theory

Introduction

The exercises 1-3 below come in the order the material has been presented. However, the most important exercise to cover end-to-end is Exercise 2, so consider starting with that. Afterwards, you may do Exercise 3. If you strugle with integration you can look at the solution guide to 2.1 and focus on doing 2.2 on your own. Finally, you can work on Exercise 1, but note that 1.3 is surprising/tricky.

Exercise 1

A fair coin is tossed n times (where n is a given positive integer).

1. Specify the state space Ω and the probability measure P for all possible realisations of the sequence of coin tosses.
2. Let A be the event "the coin toss sequence contains both a head and a tail" and let B be the event that "there is at most one tail in the sequence". Determine $P(A)$, $P(B)$, and $P(A \cap B)$.
3. Are A and B independent events?

Exercise 2

A number X is picked uniformly at random on the interval $[0, 1]$, that is, for any $I \subseteq [0, 1]$, we have $P(X \in I) =$ length of I. We say that X is *uniformly distributed between 0 and 1* and write $X \sim \text{unif}(0, 1)$.

1. Specify the distribution function, density function, mean, and variance of X.
2. What is the probability that the first decimal of X is equal to 1.

Exercise 3

A random variable X is said to follow an *exponential distribution with parameter* $\lambda > 0$ if X has density

$$f_X(x) = \lambda \exp(-\lambda x), \qquad x > 0$$

(meaning that $f_X(x) = 0$ if $x \leq 0$).

1. Determine the distribution function and the mean of X.
2. For any numbers $s > 0$ and $t > 0$, find $P(X > t + s | X > s)$ and interpret the result.