Solutions for module 6

The Gibbs sampler

1 IQ test

1. The joint distribution of $(\underline{x}, \underline{\mu}, \mu_G, \tau_G)$ has pdf

$$\begin{aligned}
\pi(\underline{x}, \underline{\mu}, \mu_G, \tau_G) &= \pi(\underline{x}|\underline{\mu}, \mu_G, \tau_G)\pi(\underline{\mu}, \mu_G, \tau_G) \\
&= \pi(\underline{x}|\underline{\mu}, \mu_G, \tau_G)\pi(\underline{\mu}|\mu_G, \tau_G)\pi(\mu_G, \tau_G) \\
&= \left(\prod_{i=1}^{n} \pi(x_i|\mu_i)\right)\left(\prod_{i=1}^{n} \pi(\mu_i|\mu_G, \tau_G)\right)\pi(\mu_G)\pi(\tau_G) \\
&= \left(\prod_{i=1}^{n} \sqrt{\frac{\tau_i}{2\pi}} \exp(-\frac{1}{2}\tau_i(x_i - \mu_i)^2)\right)\left(\prod_{i=1}^{n} \sqrt{\frac{\tau_G}{2\pi}} \exp(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2)\right) \times \\
&\quad \left(\sqrt{\frac{\tau_0}{2\pi}} \exp(-\frac{1}{2}\tau_0(\mu_G - \mu_0)^2)\right)\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\tau_G^{\alpha-1}e^{-\tau_G/\beta}\right).
\end{aligned}$$

2. The pdf for the full conditional distribution of μ_i is

$$\begin{aligned}
&\pi(\mu_i|\mu_1, \ldots, \mu_{i-1},\mu_{i+1}, \ldots, \mu_n, \mu_G, \tau_G, \underline{x}) \\
&\qquad \propto \pi(x_i|\mu_i)\pi(\mu_i|\mu_G, \tau_G) \\
&\qquad = \sqrt{\frac{\tau_i}{2\pi}} \exp(-\frac{1}{2}\tau_i(x_i - \mu_i)^2)\sqrt{\frac{\tau_G}{2\pi}} \exp(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2) \\
&\qquad \propto \exp\left(-\frac{1}{2}(\tau_i + \tau_G)\mu_i^2 + (\tau_i x_i + \tau_G\mu_G)\mu_i\right),
\end{aligned}$$

which we recognise as the unnormalised density of a normal distributed random variable with mean $(\tau_i x_i + \tau_G\mu_G)/(\tau_i + \tau_G)$ and precision $(\tau_i + \tau_G)$. Hence the full conditional distribution of μ_i is

$$\mu_i|\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n, \mu_g\tau_g, \underline{x} \sim N\left(\frac{\tau_i x_i + \tau_G\mu_G}{\tau_i + \tau_G}, \tau_i + \tau_G\right).$$

The pdf for the full conditional distribution of μ_G is

$$\begin{aligned}
\pi(\mu_G|\underline{\mu}, \tau_G, \underline{x}) &\propto \prod_{i=1}^{n} \pi(\mu_i|\mu_G, \tau_G)\pi(\mu_G) \\
&= \left(\prod_{i=1}^{n} \sqrt{\frac{\tau_G}{2\pi}} \exp(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2)\right)\sqrt{\frac{\tau_0}{2\pi}} \exp(-\frac{1}{2}\tau_0(\mu_G - \mu_0)^2) \\
&\propto \exp(-\frac{1}{2}\tau_G \sum(\mu_i - \mu_G)^2) \exp(-\frac{1}{2}\tau_0(\mu_G - \mu_0)^2) \\
&= \exp(-\frac{1}{2}(n\tau_G + \tau_0)\mu_G^2 + (\tau_G\bar{\mu} + \tau_o\mu_0)\mu_G - \frac{1}{2}(\tau_G + \tau_0)\sum \mu_i^2).
\end{aligned}$$

Again we recognise this an unnormalised density of a normal distributed random variable obtaining

$$\mu_G|\underline{\mu}, \tau_G, \underline{x} \sim N\left(\frac{n\tau_G\bar{\mu} + \tau_o\mu_0}{n\tau_G + \tau_0}, n\tau_G + \tau_0\right).$$

Finally, the full conditional distribution for τ_G is given by

$$\begin{aligned}
\pi(\tau_G|\underline{\mu}, \mu_g, \underline{x}) &\propto \prod_{i=1}^{n} \pi(\mu_i|\mu_G, \tau_G)\pi(\tau_G) \\
&= \left(\prod_{i=1}^{n} \sqrt{\frac{\tau_G}{2\pi}} \exp(-\frac{1}{2}\tau_G(\mu_i - \mu_G)^2)\right)\tau_G^{\alpha-1}\beta^\alpha e^{-\tau_G/\beta} \\
&\propto \tau_G^{n/2+\alpha-1} \exp\left(-\tau_G(\frac{1}{2}\sum_{i=1}^{n}(\mu_i - \mu_G)^2 + 1/\beta)\right),
\end{aligned}$$

which can be recognised as the unnormalised denisty of a gamma distributted random variable, hence

$$\tau_G | \underline{\mu}, \mu_g, \underline{x} \sim Gamma\left(\frac{n}{2} + \alpha, \left(\frac{1}{2}\sum_{i=1}^{n}(\mu_i - \mu_G)^2 + 1/\beta\right)^{-1}\right).$$

3. First step in a Gibbs sampler is to chose inital values. For $\mu_1^{(0)}, \ldots, \mu_n^{(0)}$ and μ_G any values will, in principle, do. The initial value $\tau_G^{(0)}$ should be positive. A more "clever" choice could be to set $\mu_i^{(0)} = x_i$ for $i = 1, \ldots, n$, $\mu_G^{(0)} = \bar{x} \equiv \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\tau_G^{(0)} = n(\sum_{i=1}^{n}(x_i - \bar{x})^2)^{-1}$.

   A Gibbs sampler could then proceed as follows

   `For t = 1 ...T do`
   - `For i = 1 ...n do`
     - `Generate` $\mu_i^{(t)}$ `from` $N\left(\frac{\tau x_i + \tau_G^{(t-1)}\mu_G^{(t-1)}}{\tau^{(t-1)} + \tau_G^{(t-1)}}, \tau^{(t-1)} + \tau_G^{(t-1)}\right)$
   - `Generate` $\mu_G^{(t)}$ `from` $N\left(\frac{n\tau_G^{(t-1)}\bar{\mu}^{(t)} + \tau_0\mu_0}{n\tau_G^{(t-1)} + \tau_0}, n\tau_G^{(t-1)} + \tau_0\right)$
   - `Generate` $\tau_G^{(t)}$ `from` $Gamma\left(\frac{n}{2} + \alpha, \left(\frac{1}{2}\sum_{i=1}^{n}(\mu_i^{(t)} - \mu_G^{(t)})^2 + 1/\beta\right)^{-1}\right)$

2 Radiocarbon dating

We start by rewriting $\pi(\mu_1, \mu_2, \mu_3) = \mathbf{1}[0 < \mu_1]\mathbf{1}[\mu_1 < \mu_2]\mathbf{1}[\mu_2 < \mu_3]\mathbf{1}[\mu_3 < k]$, where $\mathbf{1}[\cdot]$ is the indicator function:

$$\mathbf{1}[\text{"}expression\text{"}] = \begin{cases} 1 & \text{if "}expression\text{" is true} \\ 0 & \text{otherwise.} \end{cases}$$

1. The joint posterior pdf is given by

$$\begin{aligned} \pi(\mu_1, \mu_2, \mu_3 | x_1, x_2, x_3) &\propto \pi(x_1, x_2, x_3 | \mu_1, \mu_2, \mu_3)\pi(\mu_1, \mu_2, \mu_3) \\ &= \pi(x_1|\mu_1)\pi(x_2|\mu_2)\pi(x_3|\mu_3)\pi(\mu_1, \mu_2, \mu_3) \\ &= \left(\prod_{i=1}^{3}\sqrt{\frac{\tau}{2\pi}}\exp(-\frac{1}{2}\tau(x_i - \mu_i)^2)\right) \times \\ &\qquad \mathbf{1}[0 < \mu_1]\mathbf{1}[\mu_1 < \mu_2]\mathbf{1}[\mu_2 < \mu_3]\mathbf{1}[\mu_3 < k]. \end{aligned}$$

2. The full conditional for μ_1 has pdf

$$\begin{aligned} \pi(\mu_1 | \mu_2, \mu_3, x_1, x_2, x_3) &\propto \pi(x_1|\mu_1)p(\mu_1, \mu_2, \mu_3) \\ &= \sqrt{\frac{\tau}{2\pi}}\exp(-\frac{1}{2}\tau(x_1 - \mu_1)^2) \times \\ &\qquad \mathbf{1}[0 < \mu_1]\mathbf{1}[\mu_1 < \mu_2]\mathbf{1}[\mu_2 < \mu_3]\mathbf{1}[\mu_3 < k] \\ &= \sqrt{\frac{\tau}{2\pi}}\exp(-\frac{1}{2}\tau(x_1 - \mu_1)^2)\mathbf{1}[0 < \mu_1 < \mu_2]\mathbf{1}[\mu_2 < \mu_3 < k], \end{aligned}$$

   which is the the density of a normal distribution with mean x_1 and precision τ restricted to the open interval $(0, \mu_2)$. In similar fashion we can show that $\pi(\mu_2|\mu_1, \mu_3, x_1, x_2, x_3)$ and $\pi(\mu_3|\mu_1, \mu_2, x_1, x_2, x_3)$ corresponds to normal densities restricted to some open interval.

3. To generate a sample μ_1 from $\pi(\mu_1|\mu_2, \mu_3, x_1, x_2, x_3)$ we proceeds as follows: Generate proposals μ_1 from $N(x_1, \tau)$ until $\mu_1 \in (0, \mu_2)$. When this happens return μ_1 as a sample from $\pi(\mu_1|\mu_2, \mu_3, x_1, x_2, x_3)$. This procedure is an example of so-called rejection sampling.